Analytical study on possible self-excited oscillation in S-shaped regions of pump-turbines

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Abstract: As a renewable energy resource, the pumped storage power station has great prospects for better balancing supply and demand, particularly with further development of intermittent power sources and the growing need for an intelligent electric grid. However, pump-turbines often involve problematic S-shaped regions in their machine characteristics and thus, while pumped storage may solve some problems in the grid, pump-turbine operation and control can lead to other problems including severe self-excited oscillation in the hydro-mechanical system. These operational challenges have been so severe that they have even led to some reported accidents. Based on two typical and complex water conveyance systems, including two pump-turbine system sharing a common tail tunnel and two pump-turbine systems sharing an upstream penstock, the mathematical equations for self-excited oscillation are deduced from the basic equations of pressurized pipe flow coupled to the pump-turbine's characteristics; moreover, the behaviour of a single pump-turbine system is obtained through simplification of the first analysis. An analytical study is performed and the amplitude–frequency characteristic is investigated by means of both non-linear vibration theory and its corresponding analytical solution algorithm through a multi-scale method. With a given case study in detail, the results show that, for the turbines staying in the S-shaped regions for a relatively long time, self-excited oscillation inevitably occurs with severe oscillation superposed by several oscillation modes by the use of power spectral analysis module in MATLAB, and numerical results are shown to agree well with the theoretical analysis.

Keywords: hydropower, hydraulic machinery, pump-turbine, oscillatory flow, self-excited oscillation, non-linear vibration theory

1 INTRODUCTION

In recent years, pumped storage power stations have achieved rapid development in the world and they have great prospect to contribute to the further development of wind power generation and other intermittent sources as a component of intelligent electric grid. Pump-turbines are commonly used in pure pumped storage power stations with a wide operating head. Since its complete characteristic curves may have local S-shaped regions, pump-turbines running or staying in these particular regions for a relatively long time may lead to unstable operation in some cases and even possible self-excited oscillation [**1**, **2**], similar to the possible existing phenomena in some pumping systems [**3**]. Recently, there are more reports about oscillation problems in the pumped storage power stations [**4–8**], most of which are closely associated with the unstable local

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S-shaped regions in the complete characteristic curves of pump-turbines.

Self-excited oscillations generally originate from different unstable factors in the hydro-mechanical system. Recently, the oscillation problem due to different unstable valves has been revealed in detail [**9–12**]. Erhart [**9**] and Gummer [**10**] analysed the self-excited oscillation phenomenon of a water conveyance system due to the unstable flow characteristic of downstream main valve or upstream sluice gate. Misraa *et al.* [**11**] investigated self-excited oscillation characteristic of control valve located in a water conveyance system. Ye *et al.* [**12**] analysed the self-excited oscillation induced by unstable globe valve opening and seal by means of field tests and numerical verification.

Meanwhile, previous research has also shown that the existing unstable local S-shaped region in the complete characteristic curves of pump-turbines is another important source of system instability. Klemm [**13**] outlined that the longer start-up time to idle operation at lower head was beneficial for the system to pass through the S-shaped region smoothly with the improved technology of using misaligned guide vanes to reduce the effect of the S-shaped region. Yokoyama and Shimmei [**14**] summarized the dynamic characteristics of pump-turbines including the S-shaped region and its effect on operation stability. Doerfler [**4**] investigated the oscillation phenomena in a pumped storage power station and concluded that middle oscillation mode was excited by unsteady flow in the S-shaped region. You *et al.* [**5**] pointed out that, due to the existing S-shaped region of pump-turbines of a pumped storage power station in People's Republic of China, unstable rotational speeds, difficult synchronization and grid interconnection operation, reversed negative power, or unstable no-load oscillation after load rejection inevitably occur, especially at lower head operation. Martin [**15**] predicted the occurrence of oscillations and the oscillation frequency in the instable S-shaped region of pump-turbines by a linear stability analysis. Nicolet *et al.* [**16**] properly simulated the unstable characteristics at runaway by one-dimensional approach and further explained the development of oscillation modes. Focusing on the generating mode at off-design conditions involving runaway and S-shaped region, a reduced-scale model of a low specific speed radial pump-turbine was applied to identify the onset and development of flow instabilities by Hasmatuchi *et al.* [**17**]. Furthermore, the pressure distributions including the unstable flow in S-shaped regions of pump-turbines were measured and discussed in model tests [**18**], and with the rapid developments of numerical computation including computational fluid dynamics simulation in the dynamic analysis of water turbines [**19**], the amplitude–frequency properties of pressure pulsation in the S-shaped regions of pump-turbines were revealed [**20–22**]. Such statements show the profound effect of local S-shaped regions and hint as various solutions, especially during start-up or in lower head operation, but there is still much to learn relating to the detailed analysis of possible self-excited oscillation concerning S-shaped regions of pump-turbines and hydraulic system. An improved analysis approach would be beneficial for explaining various oscillation problems and for providing correct guidance for reducing oscillations and achieving stable operation.

Therefore, it is desirable to have a more complete analysis of the hydro-mechanical system's hydraulic characteristics including the pump-turbine oscillation characteristics in S-shaped regions. The goal is to find ways of not only limiting the time a pump-turbine operates in these possible unstable areas but also permitting it to pass through these regions relatively smoothly. After analysing the criterion for self-excited oscillation and corresponding unstable regions, this article considers two typical pumped storage systems and develops their mathematical equations for self-excited oscillation from coupling the basic equations of pressurized pipe flow to the pump-turbine's characteristics. Furthermore, the analytical study of corresponding self-excited oscillation has been performed by means of both non-linear vibration theory and corresponding analytical solution algorithm – multi-scale method, and a further study on possible self-excited oscillation of a given pumped storage power station has been conducted by the use of the method of characteristics for transient flow analysis and power spectral analysis module in MATLAB.

2 CHARACTERISTIC EVALUATION OF PUMP–TURBINES

2.1 Flow characteristic of pump-turbines

The flow characteristics of pump-turbines are often described with four-quadrant diagram including a series of wicket's equal-opening curves with a runaway line, shown in Fig. 1, in which n_{11}, Q_{11}, and M_{11} are unit speed, unit flow, and unit torque, respectively. Usually, it is stable for pump-turbines running in turbine condition and pump condition with condition switch specified in Fig. 2 with two typical equal-opening curves, and all the obtained

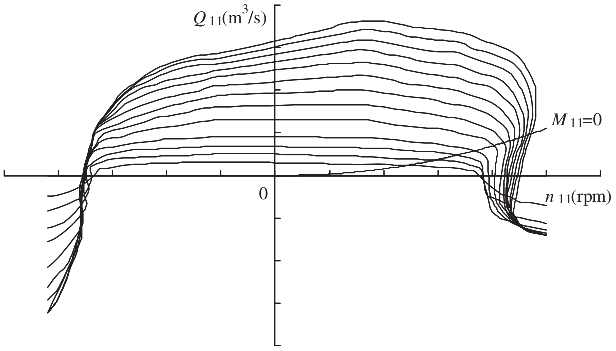

Fig. 1 Flow characteristics of pump-turbines

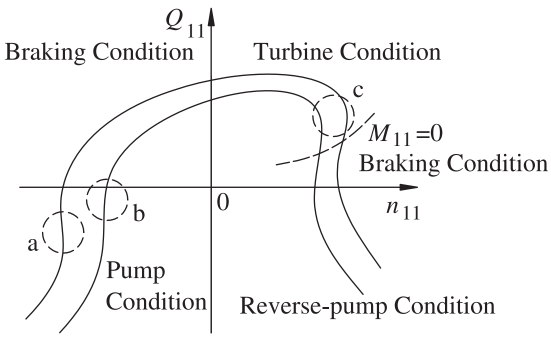

Fig. 2 Local S-shaped regions in equal-opening curves

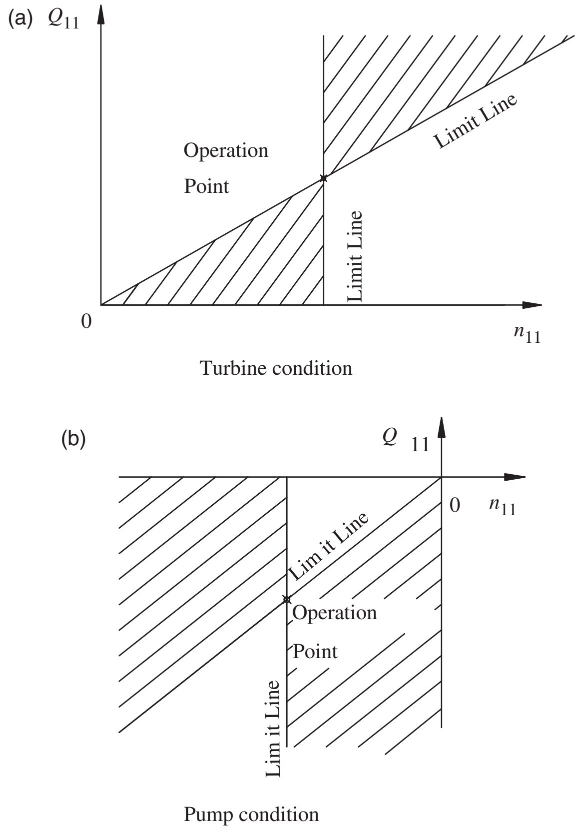

Fig. 3 Possible unstable regions of pump-turbines: (a) turbine condition; (b) pump condition

attenuation factors of the natural frequencies for the hydraulic–mechanical system are negative in a free-vibration analysis [**23**]. However, Fig. 2 also shows that there are locally steep regions in the first and third quadrants named as the S-shaped regions. If the pump-turbine operation persists in these S-shaped regions, instability will inevitably occur and may result in possible self-excited oscillation.

2.2 Criterion of S-shaped regions

For the S-shaped regions locally appearing in the complete characteristic curves of pump-turbines, the criterion for turbine condition is $\frac{\partial Q_{11}}{\partial n_{11}} > \frac{Q_{11}}{n_{11}}$, while that for pump condition $\frac{\partial Q_{11}}{\partial n_{11}} < \frac{Q_{11}}{n_{11}}$ [**24**]. According to each criterion of self-excited oscillation, the unstable regions can be illustrated with the shaded regions in Fig. 3.

In turbine condition, the mechanism for possible unstable operation is that the flow, Q_t, locally decreases as the head, H_t, increases, and it is similar to the unstable valve with the same flow characteristic; similarly, in pump condition, the mechanism for possible unstable operation is that the flow, Q_p, locally increases as the dynamic head H_p increases. The local head-flow relations are also shown in Fig. 4.

2.3 Analysis of S-shaped regions

Figure 3 shows that the unstable regions in the complete characteristic curves are just the local S-shaped regions. If the S-shaped regions are at the edges of turbine/pump condition, it is possible that a transient condition will simply encounter these regions. In contrast, the higher oscillation periods, revealed through a free-vibration analysis, are often less than 1 s and the transient process is usually intense for about 6–15 s; thus, if the unstable region is wide enough and pump-turbines linger in or near these regions, a self-excited oscillation becomes likely. This is particularly true if pump-turbine's operating point lies anywhere within the S-shaped region of pump/turbine condition. In such cases, it is important to conduct the self-excited oscillation analysis and present necessary measures to reduce the resulting oscillations.

Usually, in the flow characteristic curves of pump-turbines, there are three possible S-shaped regions which are satisfied with the given criterion including regions 'a', 'b', and 'c', indicated in Fig. 2. Region 'a' is relatively important because it is just in the normal operation region and regions 'b' and 'c' are near the

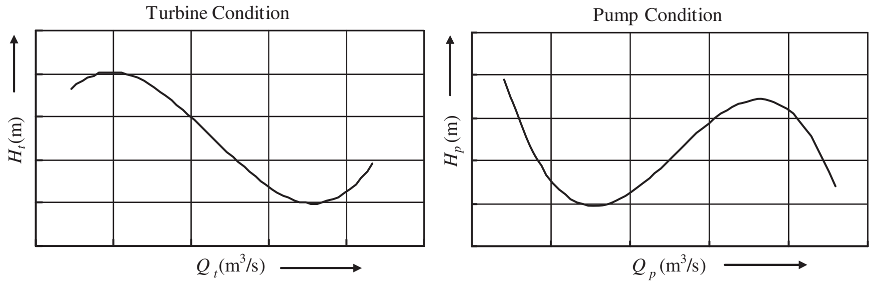

Fig. 4 Local flow characteristic in S-shaped regions

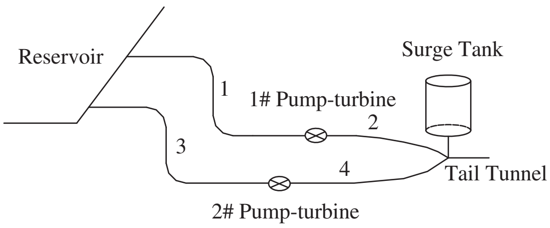

Fig. 5 Schematic diagram of two pump-turbine systems sharing a common tail tunnel

transient condition regions; yet, regions 'b' and 'c' are also the important transient regions for system start-up. Because of the analysis of regions 'b' and 'c' is similar to region 'a', the oscillation characteristics of region 'a' are discussed in detail through a case study. Although the developed self-excited oscillation obviously has adverse effect on normal operation of the pump-turbines, its effect on pump-turbines' rotational speed is often ignored in the analysis of self-excited oscillation with the consideration of their parallel operation with power grids and the relatively short time for its development.

3 DIFFERENTIAL MODELS FOR SELF-EXCITED OSCILLATION IN S-SHAPED REGIONS

3.1 Two pump-turbine systems sharing a common tail tunnel

Figure 5 shows two pump-turbine systems sharing a downstream surge tank and a common tail tunnel and the operating parameters of two slightly different branches. The mathematical equations are now established to describe the self-excited oscillation in S-shaped regions 'a' and 'b' in pump condition, and the parameters are denoted by subscript 'p'.

Considering the branch with pump-turbine 1, according to the rigid water column theory, the unsteady flow in the penstock 1 and tail branch 2 is, respectively, described by

$$\frac{L_1}{gA_1}\frac{\mathrm{d}Q_{\mathrm{p1}}}{\mathrm{d}t} = H_{u1} - H_{\mathrm{R}} - \Delta h_{\mathrm{p1}} \quad (1)$$

$$\frac{L_2}{gA_2}\frac{\mathrm{d}Q_{\mathrm{p1}}}{\mathrm{d}t} = H_{\mathrm{sd}} - H_{d1} - \Delta h_{\mathrm{p2}} \quad (2)$$

where L_j and A_j ($j=1,2$) are length and cross-sectional area of the pipe; Q_{p1} the instantaneous branch flow with pump-turbine 1 in pump condition; H_{u1} and H_{d1} the piezometric heads at the inlet and outlet of pump–turbine 1; H_R the water level of the reservoir; H_{sd} the water level of downstream surge tank; Δh_{pj} ($j=1$, 2) the head loss of each pipe in pump condition, $\Delta h_{pj} = \alpha_{pj} Q_{\mathrm{p1}}^2$; α_{pj} ($j=1$, 2) the head loss coefficient of each pipe in pump condition; and g the gravitational acceleration.

Assuming that the flow in the tail tunnel remains constant in the oscillation analysis, the continuity condition in the tank relates the surge rise in the chamber to the inflow

$$A_{\mathrm{sd}}\frac{\mathrm{d}H_{\mathrm{sd}}}{\mathrm{d}t} = Q_{\mathrm{p10}} + Q_{\mathrm{p20}} - Q_{\mathrm{p1}} - Q_{\mathrm{p2}} \quad (3)$$

where A_{sd} is the effective area of downstream surge tank; Q_{p2} the instantaneous branch flow with pump-turbine 2 in pump condition; and Q_{p10} and Q_{p20} the initial branch flows in pump condition.

After summarizing (1) and (2), then defining $\frac{L_1}{gA_1} + \frac{L_2}{gA_2} = \frac{1}{k_{\mathrm{p1}}}$ and $H_{\mathrm{p1}} = H_{u1} - H_{d1}$, the dynamic relation simplifies to

$$\frac{1}{k_{\mathrm{p1}}}\frac{\mathrm{d}Q_{\mathrm{p1}}}{\mathrm{d}t} = H_{\mathrm{sd}} + H_{\mathrm{p1}} - H_{\mathrm{R}} - \Delta h_{\mathrm{p1}} - \Delta h_{\mathrm{p2}} \quad (4)$$

where H_{p1} is the dynamic head of pump-turbine 1 in pump condition.

Further derivation of t in (4) and substituting (3) into the obtained equation yield

$$\frac{\mathrm{d}^2 Q_{\mathrm{p1}}}{\mathrm{d}t^2} = k_{\mathrm{p1}}\frac{\mathrm{d}\left(H_{\mathrm{p1}} - \Delta h_{\mathrm{p1}} - \Delta h_{\mathrm{p2}}\right)}{\mathrm{d}t} + \frac{k_{\mathrm{p1}}}{A_{\mathrm{sd}}}\left(Q_{\mathrm{p10}} + Q_{\mathrm{p20}} - Q_{\mathrm{p1}} - Q_{\mathrm{p2}}\right) \quad (5)$$

Let $H_{\mathrm{p1}} = a_1 Q_{\mathrm{p1}}^3 + b_1 Q_{\mathrm{p1}}^2 + c_1 Q_{\mathrm{p1}} + d_1$, in which a_1, b_1, c_1, and d_1 are constant coefficients approximately deduced from the flow characteristic in the local S-shaped region, and introduce incremental representation with $Q_{p1}=Q_{p10}+q_{p1}$, $Q_{p2}=Q_{p20}+q_{p2}$, in which q_{pi} $(i=1,\ 2)$ is the oscillation flow of each branch in pump condition, then

$$\begin{aligned}f(Q_{p1}) &= H_{p1}-\Delta h_{p1}-\Delta h_{p2}\\ &= a_1Q_{p1}^3+b_1Q_{p1}^2+c_1Q_{p1}+d_1-(\alpha_{p1}+\alpha_{p2})Q_{p1}^2\\ &= a_1(Q_{p10}+q_{p1})^3+(b_1-\alpha_{p1}-\alpha_{p2})(Q_{p10}+q_{p1})^2\\ &\quad + c_1(Q_{p10}+q_{p1})+d_1\end{aligned} \tag{6}$$

Equation (6) is expanded and made derivation of t

$$\begin{aligned}\frac{\mathrm{d}f(Q_{p1})}{\mathrm{d}t} &= \Big\{\Big[3a_1Q_{p10}^2+2(b_1-\alpha_{p1}-\alpha_{p2})Q_{p10}+c_1\Big]\\ &\quad +2[3a_1Q_{p10}+(b_1-\alpha_{p1}-\alpha_{p2})]q_{p1}+3a_1q_{p1}^2\Big\}\frac{\mathrm{d}q_{p1}}{\mathrm{d}t}\end{aligned} \tag{7}$$

Substituting (7) into (5) and defining the following parameters

$$k_{p1}\Big[3a_1Q_{p10}^2+2(b_1-\alpha_{p1}-\alpha_{p2})Q_{p10}+c_1\Big]=\varepsilon_{p1},$$

$$\frac{2[3a_1Q_{p10}+(b_1-\alpha_{p1}-\alpha_{p2})]}{3a_1Q_{p10}^2+2(b_1-\alpha_{p1}-\alpha_{p2})Q_{p10}+c_1}=\sigma_{p1},$$

$$\frac{3a_1}{3a_1Q_{p10}^2+2(b_1-\alpha_{p1}-\alpha_{p2})Q_{p10}+c_1}=\delta_{p1},$$

$$\frac{k_{p1}}{A_{sd}}=\omega_{10}^2$$

Rearranging, the equation becomes

$$\frac{\mathrm{d}^2q_{p1}}{\mathrm{d}t^2}-\varepsilon_{p1}\frac{\mathrm{d}q_{p1}}{\mathrm{d}t}\Big(1+\sigma_{p1}q_{p1}+\delta_{p1}q_{p1}^2\Big)+\omega_{10}^2q_{p1}=-\omega_{10}^2q_{p2} \tag{8}$$

Similarly, with the same definition of parameters k_{p2}, ε_{p2}, σ_{p2}, δ_{p2}, and ω_{20} for the branch of pump-turbine 2, the similar oscillation equation can be deduced.

$$\frac{\mathrm{d}^2q_{p2}}{\mathrm{d}t^2}-\varepsilon_{p2}\frac{\mathrm{d}q_{p2}}{\mathrm{d}t}\Big(1+\sigma_{p2}q_{p2}+\delta_{p2}q_{p2}^2\Big)+\omega_{20}^2q_{p2}=-\omega_{20}^2q_{p1} \tag{9}$$

Equations (8) and (9) are the self-excited oscillation equations for these two branches. Because the two branches converge at a bifurcation point with downstream surge tank, coupling terms inevitably appear in these two equations; in each equation, for one branch, there is only a first-order coupling term concerned with another branch; therefore, equations (8) and (9) actually describe a linear resistor coupling system with two degrees of freedom and close natural frequencies.

Supposing that the parameters of two branches shown in Fig. 5 are equal, with $q_{p1}=q_{p2}=q_p$, $\varepsilon_{p1}=\varepsilon_{p2}=\varepsilon_p$, $\sigma_{p1}=\sigma_{p2}=\sigma_p$, $\delta_{p1}=\delta_{p2}=\delta_p$, and $\omega_{10}=\omega_{20}=\omega_0$, equations (8) and (9) can be simplified into a single self-excited oscillation branch for study

$$\frac{\mathrm{d}^2q_p}{\mathrm{d}t^2}-\varepsilon_p\frac{\mathrm{d}q_p}{\mathrm{d}t}\Big(1+\sigma_pq_p+\delta_pq_p^2\Big)+2\omega_0^2q_p=0 \tag{10}$$

Equation (10) is the self-excited oscillation equation to describe the S-shaped property in pump condition for two symmetrical branches whose natural frequency is 1.414 times the natural frequency ω_0 of each branch.

Also, the mathematical equations for self-excited oscillation of turbine condition in the S-shaped region can be deduced and analysed similar to that of pump condition.

3.2 Two pump-turbine systems sharing a common penstock

Figure 6 shows two pump-turbine systems sharing a penstock and an upstream surge tank. Similarly, its mathematical equations can be deduced for describing the self-excited oscillation in the S-shaped region ‘c’ in turbine condition.

Similar to the derivation for the two pump-turbine systems sharing a common tail tunnel, based on the equations for the rigid column flow in the penstock 2 and 3 and tail branch 5, along with the continuity condition in the upstream surge tank for the surge rise in the chamber to the inflow, the oscillation equation for each branch is derived.

Assuming that the tail tunnel flow remains constant in the oscillation analysis, defining $k_2=gA_2/L_2$, $k_{t1}=1/(L_3/gA_3+L_5/gA_5)$, $k_{t2}=1/(L_4/gA_4+L_6/gA_6)$, $R_{t1}=1+(k_2+k_{t2})/k_{t1}$, and $R_{t2}=1+(k_2+k_{t1})/k_{t2}$ with the given pipe parameters and introducing the coefficients

$$\varepsilon_{ti}=-k_{ti}[3a_2Q_{ti0}^2+2(b_2+\alpha_{t3}+\alpha_{t5})Q_{ti0}+c_2]$$

$$\sigma_{ti}=\frac{2[3a_2Q_{ti0}+(b_2+\alpha_{t3}+\alpha_{t5})]}{3a_2Q_{ti0}^2+2(b_2+\alpha_{t3}+\alpha_{t5})Q_{ti0}+c_2}\ \text{and}$$

$$\delta_{ti}=\frac{3a_2}{3a_2Q_{ti0}^2+2(b_2+\alpha_{t3}+\alpha_{t5})Q_{ti0}+c_2}$$

with the approximate head-flow relation in the given S-shaped region, $H_{ti}=a_2Q_{ti}^3+b_2Q_{ti}^2+c_2Q_{ti}+d_2$ $(i=1,\ 2)$, in which a_2, b_2, c_2, and d_2 are constant coefficients approximately deduced from the flow

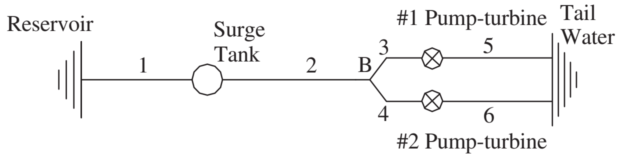

Fig. 6 Schematic diagram of two pump-turbine systems sharing a common penstock

characteristic in the local S-shaped region, the oscillation equation for the branch with pump-turbine 1 can be deduced and expressed as follows.

$$\begin{aligned}&\frac{d^2 q_{t1}}{dt^2}-\frac{(k_2+k_{t2})\varepsilon_{t1}}{k_{t1}R_{t1}}\left(1+\sigma_{t1}q_{t1}+\delta_{t1}q_{t1}^2\right)\frac{dq_{t1}}{dt}\\&+\frac{2k_2\alpha_{t2}}{R_{t1}}\left(Q_{t20}+q_{t1}+q_{t2}\right)\frac{dq_{t1}}{dt}+\frac{k_2}{R_{t1}A_{su}}q_{t1}\\&=-\frac{k_2}{R_{t1}A_{su}}q_{t2}-\frac{\varepsilon_{t2}}{R_{t1}}\left(1+\sigma_{t2}q_{t2}+\delta_{t2}q_{t2}^2\right)\frac{dq_{t2}}{dt}\\&-\frac{2k_2\alpha_{t2}}{R_{t1}}\left(Q_{t20}+q_{t1}+q_{t2}\right)\frac{dq_{t2}}{dt}\end{aligned}\tag{11}$$

where L_j and A_j ($j=1, 2,\ldots, 6$) are length and cross-sectional area of the pipe, respectively; α_{tj} ($j=1, 2, \ldots, 6$) the head loss coefficient of each pipe in turbine condition; Q_{ti} ($i=1, 2$) the instantaneous branch flow in the turbine condition; Q_{ti0} ($i=1, 2$) the initial branch flow in the turbine condition; H_{ti} ($i=1, 2$) the dynamic head of each pump-turbine in turbine condition; q_{ti} ($i=1, 2$) the oscillation flow of each branch in turbine condition; and A_{su} the effective area of upstream surge tank.

Equation (11) describes a linear resistor coupling and non-linear capacitance coupling system with two degrees of freedom. The similar derivation can be realized for the branch pump-turbine 2.

In equation (11), assuming the length of penstock 2 tending to be zero, the parameter k_2 is close to infinite value; then, the following limit values are approved.

$$\begin{aligned}&\lim_{k_2\to\infty}\frac{k_2+k_{t2}}{R_{t1}}=k_{t1},\ \lim_{k_2\to\infty}\frac{k_2}{R_{t1}}\\&=k_{t1},\ \lim_{k_2\to\infty}\frac{k_{t2}}{R_{t1}}=0,\ \lim_{L_2\to 0}\alpha_2=0\end{aligned}$$

Equation (11) can be simplified into

$$\begin{aligned}&\frac{d^2 q_{t1}}{dt^2}-\varepsilon_{t1}\frac{dq_{t1}}{dt}\left(1+\sigma_{t1}q_{t1}+\delta_{t1}q_{t1}^2\right)+\frac{k_{t1}}{A_{su}}q_{t1}\\&=-\frac{k_{t1}}{A_{su}}q_{t2}\end{aligned}\tag{12}$$

Supposing that the two branches shown in Fig. 6 are in symmetry, then $q_{t1}=q_{t2}=q_t$, $\varepsilon_{t1}=\varepsilon_{t2}=\varepsilon_t$, $\sigma_{t1}=\sigma_{t2}=\sigma_t$, $\delta_{t1}=\delta_{t2}=\delta_t$, $k_{t1}=k_{t2}=k_t$, and $R_{t1}=R_{t2}=R_t$, equation (11) results in

$$\begin{aligned}&\frac{d^2 q_t}{dt^2}-\frac{k_2\varepsilon_t}{k_tR_t}\left(1+\sigma_tq_t+\delta_tq_t^2\right)\\&\frac{dq_t}{dt}+\frac{4k_2\alpha_{t2}}{R_t}\left(Q_{t20}+2q_t\right)\frac{dq_t}{dt}+\frac{2k_2}{R_tA_{su}}q_t=0\end{aligned}\tag{13}$$

Equation (12) is similar in form to (8) or (9) for two pump-turbine systems sharing a common tail tunnel and equation (13) is close to (10) for two pump-turbine systems sharing a common tail tunnel with symmetrical branches.

3.3 Single pump-turbine system

Considering a certain branch with zero flow of two pump-turbine systems sharing a common tail tunnel, as shown in Fig. 5, it can be equivalent to a single pipe and single pump-turbine system. With $q_{p2}=0$, $q_{p1}=q_p$, $\varepsilon_{p1}=\varepsilon_p$, $\sigma_{p1}=\sigma_p$, $\delta_{p1}=\delta_p$, and $\omega_{10}=\omega_0$, equation (8) is simplified into mathematical equation for self-excited oscillation in pump condition of this simple system with downstream surge tank, that is

$$\frac{d^2 q_p}{dt^2}-\varepsilon_p\frac{dq_p}{dt}\left(1+\sigma_pq_p+\delta_pq_p^2\right)+\omega_0^2q_p=0\tag{14}$$

Similarly, the mathematical equation for self-excited oscillation in turbine condition can be established with the subscript ‘p’ of all the parameters replaced by subscript ‘t’ to express the corresponding parameters of turbine condition.

4 ANALYTICAL ANALYSIS

4.1 Analytical algorithm of non-linear oscillation equation

As the self-excited oscillation equation of pump condition is formally identical with turbine condition for single pump-turbine system or different two pump-turbine systems with symmetrical branches, the uniform equation can be written as

$$\frac{d^2x}{dt^2}-\varepsilon\frac{dx}{dt}\left(1+\sigma x+\delta x^2\right)+\omega_0^2x=0\tag{15}$$

Equation (15) is a typical non-linear oscillation equation, including damping term with the coefficients ε, σ, and δ, and the resilience term. Generally, ε is a smaller positive constant and δ a negative constant with smaller absolute value; therefore, equation (15) has described a self-excited oscillation of weakly non-linear autonomous system. Most importantly, the smaller positive constant ε means the possibility of negative damping and mainly represents the instability in the S-shaped regions. Only if larger flow disturbance occurs, the self-excited oscillations will not occur in advance because of positive damping; otherwise, the self-excited oscillations will inevitably appear due to negative damping for relatively small flow disturbance. Furthermore, equation (15) can be simplified into the typical van der Pol equation for $\sigma=0$; therefore, the instable oscillations in the S-shaped regions of pump-turbines should have the similar amplitude–frequency property of the van der Pol system [**25**, **26**]. Equation (15) can also be solved with reasonable analytical solution algorithm, such as multi-scale method [25, 26].

4.2 Single pump-turbine system

Based on the multi-scale method, the two degree approximate analytical solution of (15) can be derived and abbreviated as

$$x(t) = E\cos(\omega_0 t + \theta) + \sum_{k=2}^{5} E_{mk}\sin(k\omega_0 t + \varphi_k) + \frac{\varepsilon^2\sigma E^2}{8\omega_0^2}\left(4 + \delta E^2\right) \quad (16)$$

where E is the amplitude of first-order oscillation, $E = \frac{2}{\sqrt{(4/E_0^2+\delta)e^{-\varepsilon t} - \delta}}$; E_0 the initial amplitude of the disturbance; E_{mk} ($k=2, 3, 4, 5$) the amplitude of the second to fifth oscillations; ω_0 the first-order natural angular frequency of this system; θ the phase angle of the first oscillation; and φ_k ($k=2, 3, 4, 5$) the phase angle of the second to fifth oscillations.

In (16), the effects of three coefficients, ε, σ, and δ, can be further explained. The smaller positive constant ε and negative constant δ are the basic premises for the development of a sustained equal-amplitude oscillation as the time increases, and the constant σ mainly contributes to the bias part of the amplitude along with ε and δ. It can also be analysed from (16) that, when $t \to \infty$ and $\delta < 0$, as long as $E_0 \neq 0$, the amplitude E will tend to a steady value $2/\sqrt{-\delta}$, which is independent of initial conditions including the value of E_0.

4.3 Two pump-turbine systems

With the definition of $q_{p1} \equiv x_1$, $q_{p2} \equiv x_2$, $\varepsilon = \varepsilon_{p1}$, $\varepsilon_{p2} = k\varepsilon_{p1}$, $-\omega_{i0}^2 = \varepsilon\gamma_i$, $\sigma_{pi} = \sigma_i$, and $\delta_{pi} = \delta_i$ ($i=1, 2$), the coupling equations (8) and (9) can be rewritten as

$$\frac{d^2x_1}{dt^2} - \varepsilon\frac{dx_1}{dt}\left(1 + \sigma_1 x_1 + \delta_1 x_1^2\right) + \omega_{10}^2 x_1 = \varepsilon\gamma_1 x_2 \quad (17)$$

$$\frac{d^2x_2}{dt^2} - \varepsilon k\frac{dx_2}{dt}\left(1 + \sigma_2 x_2 + \delta_2 x_2^2\right) + \omega_{20}^2 x_2 = \varepsilon\gamma_2 x_1 \quad (18)$$

Furthermore, the approximate analytic solution of coupling equations (17) and (18) can be derived by the use of a multi-scale method [26], and in the derivation process, it is easy to gain the amplitude analytical equations

$$\frac{dE_1}{dt} = \frac{\varepsilon}{2}E_1\left(1 + \frac{\delta_1 E_1^2}{4}\right) + \frac{\varepsilon\gamma_1 E_2}{2\omega_{10}}\sin(\theta_2 - \theta_1) \quad (19)$$

$$\frac{dE_2}{dt} = \frac{\varepsilon}{2}kE_2\left(1 + \frac{\delta_2 E_2^2}{4}\right) - \frac{\varepsilon\gamma_2 E_1}{2\omega_{20}}\sin(\theta_2 - \theta_1) \quad (20)$$

As self-excited oscillation of these two branches is synchronous, the amplitude E_i ($i=1, 2$) and phase angle θ_i ($i=1, 2$) for each branch satisfy the equality

$$\frac{dE_1}{dt} = \frac{dE_2}{dt} = \frac{d\theta_1}{dt} = \frac{d\theta_2}{dt} = 0 \quad (21)$$

Substituting into equations (19) and (20) and by rearrangement, the following equality is obtained.

$$\gamma_2\omega_{10}E_1^2\left(1 + \frac{\delta_1}{4}E_1^2\right) + k\gamma_1\omega_{20}E_2^2\left(1 + \frac{\delta_2}{4}E_2^2\right) = 0 \quad (22)$$

Equation (22) is a necessary condition of the amplitudes for synchronous oscillation of these two branches. For further consideration, if these two self-excited oscillation branches are basically or completely symmetric, that is approximately considering $\omega_{10} = \omega_{20}$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2 = \delta$, and $k=1$, the equality $E_1 = E_2 = E = 2/\sqrt{-\delta}$ can be obtained which is identical with the conclusion of analytical analysis for self-excited oscillation of single pump-turbine system.

5 CASE ANALYSIS

5.1 Amplitude–frequency property of possible self-excited oscillation

A given pumped storage power station with 126.0 m rated head consists of two approximately symmetrical branches, including the penstock with length 450.0 m and diameter 5.7 m, the tail branch with length 100.0 m and diameter 6.2 m, and a downstream surge tank with area 233.4 m^2, shown in Fig. 5. With the initial flow of the pump-turbine of 60 m^3/s both in pump and turbine conditions, in Fig. 4, the head-flow relations in the local S-shaped regions are approximately defined as

$$H_t = 0.000\,09Q_t^3 - 0.0102Q_t^2 - 0.0195Q_t + 88.266$$

(for the S-shaped region ‘c’) and

$$H_p = -0.000\,64Q_p^3 + 0.116\,55Q_p^2 - 6.8174Q_p + 282.95$$

(for the S-shaped region ‘a’)

For the pump-turbines running or staying in the S-shaped region of pump condition for a relatively long time, with the given total head loss coefficient of each branch, $\alpha_{p1} + \alpha_{p2} = 0.0002$, by the use of hydraulic transients analysis based on the method of characteristics [23], the oscillation curves of the pump-turbine’s dynamic head and flow are obtained and shown in Fig. 7.

This result indicates that the oscillation amplitudes of dynamic head and flow of the pump-turbine gradually increase until a sustained equal-amplitude oscillation appears. The maximum oscillation amplitude of flow is 23.6 m^3/s which is identical with

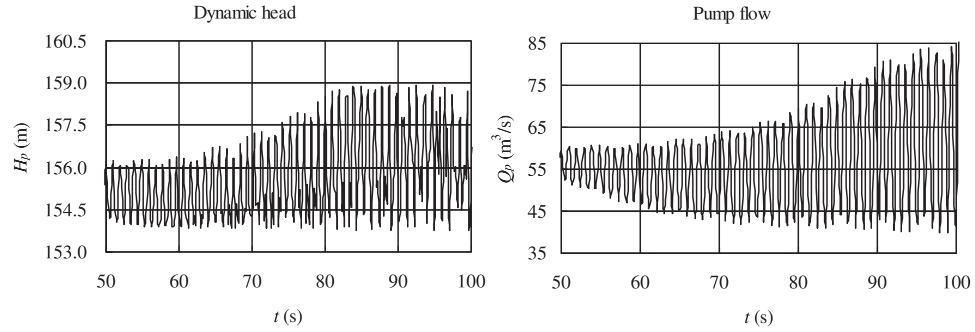

Fig. 7 Oscillation curves of self-excited oscillation

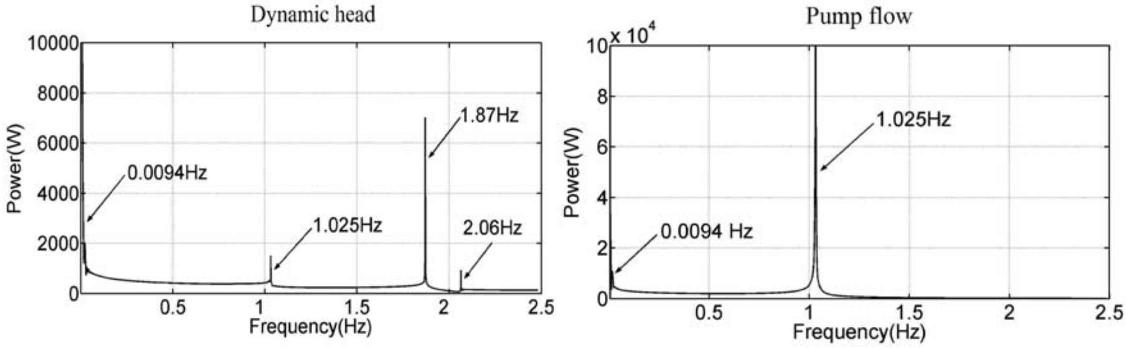

Fig. 8 Power spectrums for self-excited oscillation

the theoretical analytical value, 23.93 m^3/s, obtained in $E = 2/\sqrt{-\delta}$ with $\delta = -0.006\,98$ corresponding to $\alpha_{p1} + \alpha_{p2} = 0.0002$, and close to the 40 per cent of initial flow, while the oscillation amplitude of dynamic head is only 2.5 m. This is consistent with flow characteristic in the S-shaped region of pump condition. In addition, the computed oscillation amplitude of pump-turbine's inlet pressure is 63.5 m, close to the 50 per cent of pump-turbine's rated head. These larger oscillation amplitudes may exceed the allowable value.

Based on the above oscillation curves, further analysis is conducted by the use of a power spectral analysis module in MATLAB, which is a classical method to evaluate the variation of signal power with different frequencies. Figure 8 shows the power spectrums based on dynamic curves of dynamic head and flow of the pump-turbine.

Analysis shows several important results:

1. Self-excited oscillation is superposed by multi-order oscillations and there exists a low frequency oscillation zone with dominant frequency, $f = 0.0094$ Hz (dominant angular frequency $\omega_0 = 0.059$ rad/s) and approaches 1.414 times the theoretical value of each branch's natural angular frequency, $\omega_0 = 0.045$ rad/s, which can be obtained in $\omega_0^2 = \frac{k_{p1}}{A_{sd}}$ with $k_{p1} = 0.468$ and $A_{sd} = 233.4$ m^2 for this case.
2. The oscillation of dynamic head appears to have three higher order oscillation modes, while that of flow is only one in the same frequency range, and all the oscillation frequencies are approximately regarded as integral times the natural frequency ω_0. These oscillation frequencies represent the typical characteristics of non-linear oscillations similar to the van der Pol system and also have obvious correlation with the pipe properties because water elasticity is considered in the method of characteristics. Considering the complexity of actual damping and pump-turbine's characteristics, the spectral analysis of self-excited oscillation is identical with analytical analysis.

5.2 Further sensitivity analysis

Because the S-shaped regions in the characteristic curves of pump-turbines usually have some uncertain factors and it is rather difficult to illustrate these characteristic in detail by experimental research or numerical simulation, two different head-flow relations with gentler slopes in the S-shaped region of pump condition relative to the given case in Fig. 9 are assumed.

Further sensitivity analysis is carried out with different total head loss coefficients ($\alpha_{p1} + \alpha_{p2}$) in (6) and presented in Table 1, in which the values

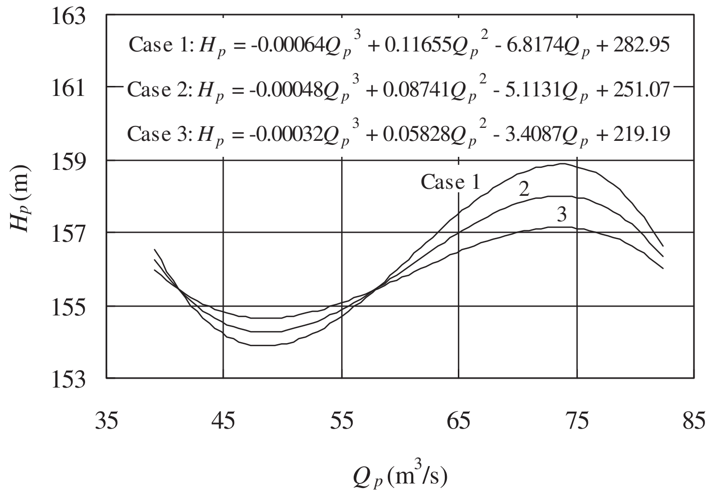

Fig. 9 Head-flow relations for different cases

Table 1 Theoretical oscillation amplitudes

	$\alpha_{p1}+\alpha_{p2}$		
Cases	0.0	0.0002	0.0004
1	24.94	23.93 (23.60)	22.86
2	24.94	23.58 (23.44)	22.11
3	24.94	22.86 (23.01)	20.54

within parentheses are computed oscillation amplitudes by the method of characteristics.

The results indicate that the oscillation amplitudes are equal for $\alpha_{p1}+\alpha_{p2}=0$ and with the increasing of the total head loss coefficient, the oscillation amplitudes are all reduced for different cases; for the larger head loss coefficient, the reduction of the oscillation amplitude becomes more obvious with the improvement of the S-shaped property. Furthermore, the computed oscillation amplitudes agree well with the theoretical results. Therefore, because of the real uncertainty in the S-shaped regions, a reasonable safety margin should be used in analysing a region of instability. It is also proved that different head-flow relations in the S-shaped regions and head loss coefficient have no obvious effect on the oscillation frequencies.

In addition, a sensitivity analysis of branch pipe length is also conducted. For the two branches with different penstock lengths, for example, 450 and 500 m, each branch has its own natural frequency and corresponding oscillation curves characterized by different frequencies according to its length. The resulting primary analytical frequencies are 0.0094 and 1.025 Hz for the branch with 450 m penstock, while 0.0089 and 0.882 Hz for the branch with 500 m penstock. As the two branches are of equivalent lengths, the severe coupling hydraulic resonance inevitably happens with the equal natural frequencies of the two branches. If the penstock lengths of the two branches both vary from 450 to 500 m, similar to Fig. 7, the hydraulic oscillations are computed out with two main oscillation frequencies 0.0089 and 0.882 Hz, and it takes relatively long time to develop into a sustained equal-amplitude oscillation, which also agrees well with the analytical result of the formula of E with the decreasing ε. Meanwhile, further analysis also shows that the variation of oscillation frequencies with the change of penstock lengths basically has no influence on the oscillation amplitude.

6 CONCLUSIONS

In pumped storage power station, the complete characteristic curves of pump-turbines may locally have S-shaped regions, and pump-turbines should pass through these regions smoothly during operation and control actions. Otherwise, severe self-excited oscillation and instabilities may occur.

Combined with the basic equations of pressurized pipelines, the mathematical equations for self-excited oscillation of two pump-turbine systems sharing a downstream surge tank or an upstream penstock were established, and then, the equation for two pump-turbine systems with symmetrical branches and single pump-turbine system were derived by further simplification. All are typical non-linear oscillation equations of weakly non-linear autonomous system. Furthermore, by the use of non-linear vibration theory and approximate analytical method – multi-scale method, the approximate analytical solution was obtained. With further amplitude–frequency analysis, it is shown that all important coefficients, ε, σ, and δ, contribute to the different parts of the analytical formula of the oscillation amplitude. Most importantly, as the time increases, the basic premises for the development of a sustained equal-amplitude oscillation mainly include that ε is a smaller positive constant and δ a negative constant in equation(15).

With a given case study, by means of the method of characteristics in time domain and power spectral analysis in frequency domain, the results indicate that, as the pump-turbines running or staying in the S-shaped regions for a relatively long time, self-excited oscillations inevitably occur with severe hydraulic oscillation superposed by several oscillation modes, and this numerical results agree well with the theoretical analysis. Further sensitivity analysis also reveals the effect of the uncertain factors in S-shaped regions, as well as head loss coefficients on the amplitude–frequency properties of possible self-excited oscillation, and the variation of oscillation frequencies with the change of penstock lengths basically has no influence on the oscillation amplitude.

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APPENDIX

Notation

A	cross-sectional area of pipe
A_{sd}	effective area of downstream surge tank
A_{su}	effective area of upstream surge tank
E	amplitude of first-order oscillation
E_0	initial amplitude of the disturbance
E_m	amplitude of the second to fifth oscillations
g	gravitational acceleration
H	dynamic head of the pump-turbine
H_d	piezometric head at the outlet of the pump-turbine
H_R	water level of the reservoir
H_{sd}	water level of downstream surge tank
H_u	piezometric head at the inlet of the pump-turbine
L	pipe length
M_{11}	unit torque
n_{11}	unit speed
q	oscillation flow of each branch
Q	instantaneous branch flow
Q_{11}	unit flow
Q_0	initial branch flow
α	head loss coefficient of the pipe
δ	coefficient of the second-order item in the damping term
Δh	head loss of the pipe
ε	coefficient of the damping term
θ	phase angle of the first oscillation
σ	coefficient of the first-order item in the damping term
φ	phase angle of the second to fifth oscillations
ω_0	first-order natural angular frequency

Subscripts

i	number of the pump-turbines or the branches
j	number of pipe sections
k	oscillation order
p	pump condition
t	turbine condition