

# MODELING LOW VELOCITY/HIGH DISPERSION FLOW IN WATER DISTRIBUTION SYSTEMS

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**ABSTRACT:** Under most circumstances, dispersive processes can be neglected in water distribution systems due to high velocities and low dispersion coefficients that ensure that advective transport dominates constituent spreading. However, during periods of low flow, dispersive effects may become important if the velocities are significantly decreased (e.g., during the night). Thus, the question that arises during a 24-h simulation is whether dispersive processes may dominate the transport of chlorine in a water distribution system. Given that advective transport models (e.g., EPANET, PICCOLO, NET, and DWQM) cannot account for dispersive transport, it is likely that these models would underpredict the required concentration of chlorine at locations behind the advective front and overpredict the required concentration at locations in advance of the advective front. Therefore, when is a purely advective transport model an appropriate solution for contaminant transport in a water distribution system? Two constraining equations are presented to aid the user in assessing the applicability of an advective transport model.

## INTRODUCTION

An adequate free-chlorine residual is critical for ensuring satisfactory disinfection of potable water supplies, but overchlorination is also undesirable. Trihalomethanes are often produced when high concentrations of free chlorine residual are maintained for a significant length of time. Since sustained high concentrations of free-chlorine residual increase the likelihood of trihalomethane formation, it is imperative that only the appropriate chlorine dosage be released into the system at a given time and location. Advective transport models (e.g., EPANET, PICCOLO, NET, and DWQM) may be used to determine these dosages, based on computed levels of chlorine decay.

Given the inability of advective transport models to account for dispersive transport, it is conceivable that required chlorine dosages will be incorrectly estimated by the models when velocities in a water distribution system are quite low. To examine the applicability of such models to low velocity/high dispersion flow, an advective transport model was compared to a closed form solution of the advection-dispersion equation.

## ADVECTION-DISPERSION EQUATION

The transport process has been greatly simplified in advective transport models by neglecting the contribution of dispersion to axial spreading. That is, the constituent concentration  $C_i(x, t)$  within a pipe segment  $i$  is obtained from the solution of the following conservation of mass equation:

$$\partial C_i(x, t)/\partial t + u_i \partial C_i(x, t)/\partial x + KC_i(x, t) = 0 \quad (1)$$

where  $x$  = longitudinal space dimension;  $t$  = time;  $u_i$  = mean velocity of flow in pipe segment  $i$ ; and  $K$  = coefficient of concentration decay rate. The overall decay coefficient was defined according to Rossman (1993) as

$$K = k_b + k_w k_f / R_H (k_w + k_f) \quad (2)$$

where  $k_b$  = bulk flow decay constant;  $k_w$  = wall decay constant;  $R_H$  = hydraulic radius; and  $k_f$  = mass transfer coefficient. For

the growth of a constituent,  $K$  should be specified as a negative value. The closed-form solution of (1) is

$$C_i(x, t + \Delta t) = C_i(x - u_i \Delta t, t) e^{-K \Delta t} \quad (3)$$

where  $\Delta t$  = the time step (Rossman et al. 1993).

If the effect of dispersion on the axial spreading of a constituent is modeled, (1) becomes

$$\begin{aligned} \partial C_i(x, t)/\partial t + u_i [\partial C_i(x, t)/\partial x] \\ - D [\partial^2 C_i(x, t)/\partial x^2] + KC_i(x, t) = 0 \end{aligned} \quad (4)$$

where  $D$  = coefficient of longitudinal dispersion. Since a direct comparison of (1) and (4) is difficult, it is desirable to develop the analytical solution of (4).

## Boundary/Initial Conditions

If it is assumed that a constant supply of constituent is available at the pipe inlet when  $t > 0$ , then the steady-state concentration of constituent in the pipe at time zero is described by

$$C_i(x, 0) = C_{00} \exp[(P_x/2)(1 - W)] \quad (5)$$

where  $P_x$  has the same form as the traditional Péclet Number [i.e.,  $(u_i x)/D$ ] but emphasizes the variability of the distance parameter  $x$ , and

$$W = (2\sqrt{D/u})\sqrt{K + u^2/4D} \quad (6)$$

$C_{00}$  in (5) is the concentration of constituent added to the pipe at  $x = 0$  to determine steady-state concentrations in the pipe prior to  $t = 0$ . In fact, (5) is obtained by solving the advection-dispersion equation with the initial condition  $C_i(x, 0) = 0$  and the partial derivative  $\partial C_i(x, t)/\partial t = 0$ . In addition, it is assumed that the constituent completely decays by the time it reaches the outlet of an infinitely long pipe [i.e.,  $C_i(\infty, t) = 0$ ]. This boundary condition is common in open channel and pipe flow problems involving contaminant transport (e.g., Fischer 1979; Li 1972). Alternatively, the exit boundary condition can be expressed as  $dC_i/dx(\infty, t) = 0$ . This alternative boundary condition gives rise to an identical analytical solution of the advection-dispersion equation for the semi-infinite domain since, in either case, the requirement of a bounded solution forces the coefficients of all exponentially growing terms to zero.

## Analytical Solution

Extended time-period analysis calculates steady-state conditions prior to the solution of the transient problem. The pro-

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cess is iterative in that the initial steady state conditions must be computed prior to transient simulation for each system-configuration. For example, utilizing a different contaminant concentration or pipe diameter results in a new steady state condition and thus, a change in the initial condition for the transient problem [i.e., (5)]. Once the boundary/initial conditions have been determined, the analytical solution of the advection-dispersion equation is

$$C_i(x, t) = \frac{C_0}{2} \exp[(P_x/2)(1 - W)] \left\{ \operatorname{erfc} \left( \frac{x - utW}{2\sqrt{Dt}} \right) + \exp(P_x W) \operatorname{erfc} \left( \frac{x + utW}{2\sqrt{Dt}} \right) \right\} + C_i(x, 0) \quad (7)$$

where  $C_0$  = concentration of constituent added to the pipe at  $x = 0$  for  $t > 0$  while

$$\frac{C_0}{2} \exp \left[ \frac{P_x}{2} (1 - W) \right] \text{ is called part 1,}$$

$$\operatorname{erfc} \left( \frac{x - utW}{2\sqrt{Dt}} \right) \text{ is called part 2,}$$

$$\operatorname{erfc} \left( \frac{x + utW}{2\sqrt{Dt}} \right) \text{ is called part 3 and}$$

$$C_i(x, 0) \text{ is called part 4.}$$

Eq. (7) can be derived by Laplace transforms and checked against the Ogata-Banks equation (e.g., Domenico and Schwartz 1990) if  $K = 0$ . Because, it is difficult to write general mathematical rules for comparing (3) and (7), it is useful to seek an engineering compromise to aid the advective transport model user.

## DISPERSION COEFFICIENT

In pursuing this engineering approach, the classical work of Taylor (1954) and Aris (1956) is helpful. Axial spreading of a constituent is the combined result of shear flow dispersion  $D_{disp}$  and diffusion  $D_{diff}$  (Aris 1956). That is

$$D = D_{disp} + D_{diff} \quad (8)$$

Each process is separately defined, based on the flow regime. For laminar pipe flow, Taylor (1953) found that shear flow dispersion of dye in water can be described by

$$D_{disp} = a^2 u^2 / 48 D_{diff} \quad (9)$$

where  $a$  = pipe radius. In addition, the diffusion process may be described by the molecular diffusion coefficient. For turbulent pipe flow, Taylor (1954) found that shear flow dispersion of dye in water can be empirically defined by

$$D_{disp} = 10.1 a u_* \quad (10)$$

where  $u_*$  = shear-stress velocity. Shear-stress velocity can be defined as  $u_* = u \sqrt{f/8}$  in which  $f$  = Darcy-Weisbach friction factor. Diffusion is described by the turbulent diffusion coefficient, but since the diffusion process is negligible in comparison to shear flow dispersion, it is commonly neglected in most engineering applications.

In the following engineering application, Taylor's dispersion coefficient for laminar pipe flow is used to illustrate the procedure by which one may assess the suitability of an advective transport model for modeling low velocity/high dispersion flows.

## APPLICATION PROBLEM

A simple pipeline, defined in Fig. 1, consists of 10 pipe segments and 11 pipe junctions (i.e., nodes 2–12). Each pipe

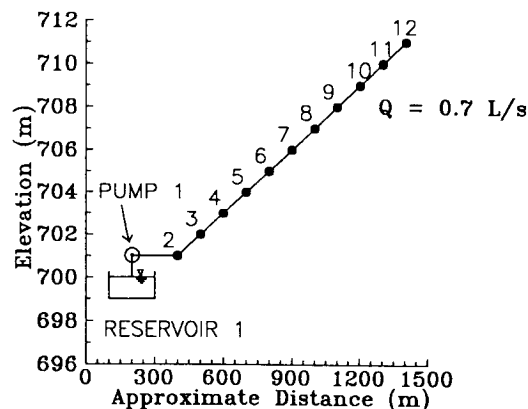


FIG. 1. Example Pipeline System

segment is 100 m long and has an inside diameter of 500 mm. A centrifugal pump (pump 1), located at the system inlet, lifts water from a constant head reservoir (reservoir 1) at an elevation of 700 m to a free discharge at an elevation of 711 m.

Chlorine, with a molecular diffusivity of  $1.21 \times 10^{-9} \text{ m}^2/\text{s}$ , is completely mixed within the storage facility to a concentration of 1.0 mg/L [i.e.,  $C_i(0, t) = 1.00 \text{ mg/L}$ ]. The pipe is free of constituent at time zero [i.e.,  $C_i(x, 0) = 0$ ]; therefore,  $C_{00} = 0$  in (7). Concentrations are measured at each node to track both the transport and decay of the constituent as it travels through the pipeline.

The flow rate in the system is a steady state 0.7 L/s, which corresponds to a velocity of 0.004 m/s, a Reynolds number ( $R$ ) of 1,748 at 19°C, and a Darcy-Weisbach friction factor of 0.0366. As this is laminar flow (i.e.,  $R < 2,000$ ), the coefficient of axial dispersion can be computed from (8) and (9) to be 13.68  $\text{m}^2/\text{s}$ . In addition, (2) can be evaluated to yield an overall decay coefficient of  $6.417 \times 10^{-6} \text{ s}^{-1}$  when the bulk flow and wall decay constants are 0.5/d and 0.3048 m/d, respectively. Due to the high dispersion coefficient and the low mean velocity, one would expect chlorine to spread well in advance of the advective front.

## ADVECTIVE TRANSPORT MODEL VERSUS ADVECTION-DISPERSION EQUATION

The application problem was analyzed using both the advective transport model and the advection-dispersion equation. The water quality time step was chosen to be 15 min while the duration of the simulation was 47 h. As Fig. 2 illustrates, the advective transport model significantly underpredicts the nodal concentrations obtained by solving the advection-dispersion equation. In fact, as Table 1 shows, the advective transport model reports a zero-level concentration at node 10 while the analytical solution reports a chlorine concentration of 0.60 mg/L. Although an increased dispersion coefficient implies a greater tendency for dispersive transport, a corresponding increase in water velocity can eliminate these effects.

## SENSITIVITY OF CONSTITUENT TRANSPORT TO DISPERSION

To determine the effect of velocity on the dispersive transport process, it is important to examine the relative importance of each portion of (7). For example, it can be shown that

$$\lim_{K \rightarrow 0} \{ C_0/2 \exp[(P_x/2)(1 - W)] \} = C_0/2 \quad (11)$$

and

$$\lim_{K \rightarrow 0} [C_i(x, 0)] = C_{00} \quad (12)$$

suggest that parts 1 and 4 of (7) can be considered independent of both the velocity  $u$  and the dispersion coefficient  $D$ .

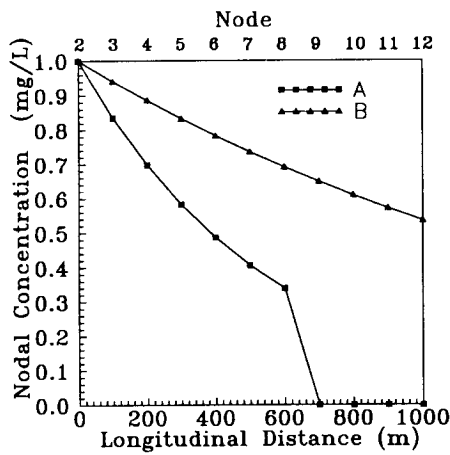


FIG. 2. Effects of Dispersion Illustrated in Terms of Nodal Concentrations at  $t = 47$  h (A = Advection Transport Model; B = Advection-Dispersion Equation)

TABLE 1. Nodal Concentration [ $C(x, t)$ ] Data at  $t = 47$  h for Advection Transport Model (A) and Advection-Dispersion Equation (B)

Node (1)	$C(x, t)$ (mg/L)	
	A (2)	B (3)
2	1.0000	1.0000
3	0.8353	0.9415
4	0.6977	0.8861
5	0.5828	0.8335
6	0.4868	0.7836
7	0.4067	0.7364
8	0.3397	0.6916
9	0.0000	0.6493
10	0.0000	0.6093
11	0.0000	0.5715
12	0.0000	0.5358

TABLE 2. Limiting Velocities above Which Advection Dominated Dispersion

Dispersion coefficient ( $m^2/s$ ) (1)	Limiting velocity (m/s) (2)
$5.0 \times 10^{-3}$	0.006
$5.0 \times 10^{-2}$	0.063
$5.0 \times 10^{-1}$	0.639
5.0	6.395
10.0	12.789

This greatly simplifies the analysis of (7) because it becomes clear that the dispersion process is primarily a function of the arguments of the complementary error functions (i.e., parts 2 and 3). As the velocity increases, the argument of part 3 increases in the positive direction while the argument of part 2 increases in the negative direction. Since the complementary error function is zero for arguments greater than 2.0 and 2.0 for arguments less than -2.0, dispersive effects will be negligible when

$$\frac{x - utW}{2\sqrt{Dt}} < -2.0 \quad \text{and} \quad \frac{x + utW}{2\sqrt{Dt}} > +2.0 \quad (13, 14)$$

With this in mind, the testing of the application problem was repeated with a number of dispersion coefficients. The velocity of flow was increased in each trial to the point where advection began to dominate dispersion. That is, to the point where the constraints [i.e., (13) and (14)] were just met. For

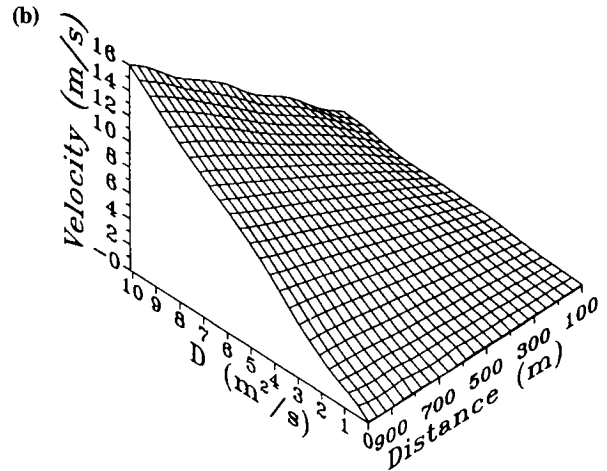
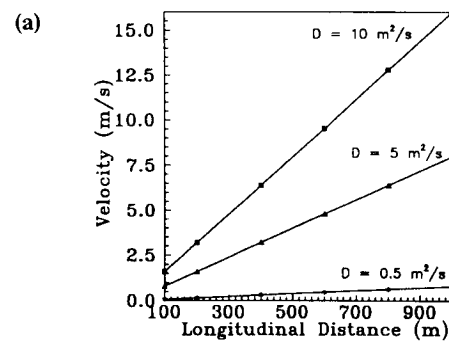


FIG. 3. Relationship Governing Sensitivity of Pipe Flow to Dispersion Expressed in Terms of Water Velocity, Dispersion Coefficient, and Axial Position

example, Table 2 lists the limiting velocities above which advection dominated dispersion at node 8.

The combination of the limiting velocity and the dispersion coefficient yields a relationship governing the sensitivity of pipe flow to dispersion at a particular location. This relationship is illustrated in Fig. 3 for a number of locations along the pipeline. If the combination of velocity and dispersion coefficient lie below the plane, dispersion will influence constituent spreading at the location in question. If the combination of velocity and dispersion coefficient lie above the plane, dispersion will not influence constituent spreading at the location in question, and the advective transport model should be suitable.

### CONCLUSIONS

The relationship illustrated in Fig. 3 is specific to the analyzed system because dispersion can vary according to the length, diameter, roughness, etc., of the pipe. Based on the preceding results, it appears that (3) is a good approximation of (7) under most engineering applications. In fact, it is rare that the velocity would be small enough to require the refinement of (7). However, if in doubt, advective transport model users can quickly assess the appropriateness of their model by evaluating (13) and (14) at important locations in their system. If (13) is greater than -2.0 and (14) is less than 2.0, dispersion is important and an advective/dispersion water quality model is recommended. If (13) is less than -2.0 and (14) is greater than 2.0, dispersion is likely to be negligible and an advective transport model should be suitable.

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## APPENDIX II. NOTATION

The following symbols are used in this paper:

- $a$  = pipe radius;  
 $C_i(x, t)$  = concentration of constituent in pipe;

- $C_{00}$  = concentration of constituent added to pipe at  $x = 0$  to determine steady state concentrations in the pipe prior to  $t = 0$ ;  
 $C_0$  = concentration of constituent added to the pipe at  $x = 0$  (i.e., at  $t > 0$ );  
 $D$  = coefficient of longitudinal dispersion;  
 $D_{diff}$  = diffusion coefficient;  
 $D_{disp}$  = shear flow dispersion coefficient;  
 $f$  = Darcy-Weisbach friction factor;  
 $K$  = concentration decay rate;  
 $k_b$  = bulk flow decay constant;  
 $k_f$  = mass transfer coefficient;  
 $k_w$  = wall decay constant;  
 $P_x = (ux)/D$ ;  
 $R$  = Reynolds number;  
 $R_H$  = hydraulic radius;  
 $t$  = time;  
 $u_i$  = mean velocity;  
 $u_*$  = shear-stress velocity;  
 $W = (2\sqrt{D}/u)\sqrt{K + u^2/4D}$ ;  
 $x$  = longitudinal space dimension; and  
 $\Delta t$  = time step.

## Subscript

- $i$  = pipe segment index.