Flexible Discretization Algorithm for Fixed-Grid MOC in Pipelines

By Bryan W. Karney\textsuperscript{1} and Mohamed S. Ghidaoui\textsuperscript{2}

\textbf{Abstract:} The problem of selecting a time step that exactly satisfies the Courant condition arises most commonly when the fixed-grid method of characteristics (MOC) is applied to multipipe systems. Traditionally, this problem has been solved by employing one of a variety of interpolation techniques or by allowing small adjustments in the value of wave speed. In this paper, the number of available strategies is increased substantially by allowing several new kinds of interpolation as well as blended combinations of these interpolation approaches with the wave-speed-adjustment technique. The new hybrid approaches include interpolation along a secondary characteristic line and minimum-point interpolation, which minimizes the distance from the interpolated point to the primary characteristic. An intuitively appealing technique involving direct adjustment of the wave path is rejected because it is potentially unstable. The entire flexible algorithm is implemented as a preprocessor step, uses memory efficiently, executes quickly, and provides a flexible tool for investigating the importance of discretization errors in pipeline systems. The properties of the new algorithm are analyzed theoretically and illustrated by example.

\textbf{Introduction}

For many years the fixed-grid method of characteristics (MOC) has been used with great success to calculate transient conditions in pipe systems and networks. In fact, for pipeline transients, this numerical procedure is noted generally for its numerical efficiency, computational accuracy, and programming simplicity (Wylie and Streeter 1993; Almeida and Koelle 1992; Chaudhry 1987). However, one difficulty that commonly arises relates to the selection of an appropriate level of discretization (or time step) to use for the analysis. The obvious trade-off is between computational speed and accuracy: in general, the smaller the time step, the longer the run time but the greater the numerical accuracy.

The challenge of selecting a time step is made difficult in pipeline systems by two conflicting constraints. First, to calculate many boundary conditions, such as obtaining the head and discharge at the junction of two or more pipes, it is necessary that the time step be common to all pipes. The second constraint arises from the nature of the MOC. If the advective terms in the governing equations are neglected (as is almost always justified), the MOC requires that ratio of the distance step $\Delta x$ to the time step $\Delta t$ be equal to the wave speed $a$ in each pipe. In other words, the Courant number $C = a\Delta t/\Delta x$ should ideally be equal to one and must not exceed one for stability reasons. For most pipeline systems, having as they do a variety of different pipes with a range of wave speeds and lengths, it is impossible to satisfy exactly the Courant requirement in all pipes with a reasonable (and common) value of $C$.

Faced with this challenge, researchers have sought for ways of relaxing the numerical constraints. Two contrasting strategies present themselves. The "method of wave-speed adjustment" changes one of the pipeline properties (usually the wave speed, though more rarely the pipe length is altered) so as to satisfy exactly the Courant condition. Despite the obvious liberties this kind of adjustment takes with the physical problem, this procedure is widely recommended in the pipeline literature (Wylie and Streeter 1993; Chaudhry 1987; Karney and McInnis 1992). The second alternative is to allow Courant numbers less than one and to interpolate between known grid points. The most common methods include linear interpolation at a fixed time level, including both space line interpolation and reach-out in space interpolation (Wiggert and Sundquist 1977), as well as interpolation at a fixed location, such as time line interpolation or reach-back in time interpolation (Goldberg and Wylie 1983). Lai (1989) combined these options to form what he calls the multimode scheme. A number of nonlinear interpolation techniques also have been proposed including the Holly-Preissmann scheme (Holly and Preissmann 1977), Holly-Preissmann with time line interpolation (Liggett and Chen 1994), and a method that uses cubic splines (Sibetileros et al. 1991).

Ghidaoui and Karney (1994) have developed the concept of an equivalent hyperbolic partial differential equation (EHDE) to analyze theoretically the numerical properties of a variety of interpolation schemes. They found that all common interpolation procedures considerably distort the original governing equations and that even interpolation procedures effectively change the wave speed. Thus, Ghidaoui and Karney concluded that it is unlikely that any discretization approach would be ideal for all pipeline systems and for all kinds of disturbances, making it improbable that the ongoing controversy about which discretization method is "best" will be resolved in the near future. However, despite the ubiquitous nature of discretization errors in complex pipeline systems, the authors contend that discretization errors are sometimes the dominant numerical error and may outweigh errors arising from nonlinear terms such as friction or minor uncertainties in the boundary conditions.

Thus, no attempt is made here to end debate about which discretization approach is best for a given problem. Nor does this paper presume to dismiss complex or higher-order methods. However, at least for the constant wave-speed cases considered here, it is logical to explore more fully simple algorithms that may either replace or be used along with more complex schemes. To this end, a flexible and efficient discretization algorithm is presented that can readily be applied to existing fixed-grid MOC codes and allows a variety of discretization choices to be selected conveniently and investigated.

More specifically, this paper uses a generalized algebraic formulation to derive a flexible class of discretization approaches for interpolating between grid points to obtain values of the dependent variables at the foot of characteristic curves.
The new approaches include interpolation along a secondary or intersecting characteristic line, minimum point interpolation that uses the shortest distance from the interpolated point to the primary characteristic, and a method of wave-path adjustment that, like wave-speed adjustment, distorts the path of propagation, but does not directly change the wave speed. In addition, any of these techniques can be blended with the method of wave-speed adjustment. The performance of the new algorithms is studied theoretically, using the equivalent EHDE approach, and practically, using a numerical example of a transient disturbance in a realistic pipeline system. Thus, the flexible interpolation algorithm is shown to be a convenient tool for investigating interpolation errors.

**NATURE OF THE INTERPOLATION PROBLEM**

Before developing a flexible interpolation algorithm, it is useful to consider the specific nature of the discretization problem when using the fixed-grid MOC to analyze a general pipeline system. First, it is emphasized that practical interpolation methods, at least for the constant wave-speed case, must be computationally efficient, because the interpolation errors can be made progressively smaller by successive reductions in the time step. Moreover, small time step may be preferable because it reduces other numerical errors, such as those arising from fluid friction or the representation of boundary conditions. However, small time steps may be computationally impractical, particularly in complex systems, because of either memory limitations or prohibitive execution times.

As an aside, it should be pointed out that short pipes may sometimes be more conveniently modeled by ignoring their compressibility effects entirely, treating these links as either incompressible or quasi-steady conduits within the larger MOC simulation (Wylie and Streeter 1993; Karney and McInnis 1992). Because the choice of time step is not independent of these links, this approach is clearly expedient from a computational point of view. In essence, the modeler ignores the wave travel time in these conduits, effectively adjusting their wave speed to infinity. However, even if the shortest possible conduits are eliminated by this approach, the discretization problem remains for all those pipes still included in the MOC simulation. It is to those pipes that remain within the MOC computational framework that the remainder of this paper is addressed.

The relationship between discretization errors in the MOC and the time step—or, equivalently, the number of reaches—is made clear in Fig. 1. This figure indicates the Courant number $C_r$, as a function of the ideal number of reaches $R = \frac{L}{\Delta t}$ in a pipeline segment of length $L$ where $R$ is a real number. But this simple relation highlights the key discretization problem, for the ideal number of reaches $R$ is almost invariably noninteger, whereas the number of reaches $N$ in each pipe must be integer. To avoid unstable Courant numbers $C_r$ with values greater than one, while simultaneously limiting interpolation distances, it is logical to select $C_r = [R]/R$ in which $[R] = N$ is the floor of $R$. However, the shift from $R$ to $N$ can represent a large proportional change, particularly when $R$ is small. For example, if $R = 1.99$, and thus $N = 1$, then the number of reaches is almost 50% smaller than that required to satisfy exactly the wave travel time.

Ghidaiou and Karney (1994) have shown that discretization errors generally grow with $|1 - C_r|$; thus it is not surprising that short pipes with small $R$ are often problematic. Yet the diminishing saw-tooth form of Fig. 1 also illustrates that a small time step with an associated large number of reaches $R$ can always be found, which reduces interpolation errors to negligible values. More precisely, the departure of the Courant number from one is bounded by $1/R$. Clearly, then, any reasonable interpolation technique using a small time step will likely perform well. However, in complex systems the computational cost for such a large number of reaches can be high because not only do execution times for a given duration of simulation increase roughly as the square of $R$, but the convergence to $C_r = 1.0$ is so irregular that modest reductions in the time step may actually temporarily increase the discretization error. Moreover, discretization approaches must generally be viable with single-reach pipes. The rather abrupt and discontinuous convergence to $C_r = 1.0$ shown in Fig. 1 underlies all discretization problems in realistic pipeline systems, greatly complicating numerical investigation.

The second (dashed) curve in Fig. 1 indicates the relative distortion in the wave speed as a function of the ideal number of reaches. Because the wave-speed—adjustment approach exactly satisfies the Courant condition, it is permissible to round off the number of reaches during the calculation for the adjusted wave speed: thus, $a/a' = [R + 0.5]/R$, where $a'$ is the adjusted wave speed. As a result, the relative adjustment in wave speed is bounded by $1/2R$, half the value associated with pure interpolation. Moreover, the fact that this technique may either increase or decrease the wave speeds means that there is a greater chance of offsetting errors between different parts of a pipeline system, thereby possibly decreasing the overall error. For particular systems, the effect of rounding may be important because it can greatly reduce the required adjustment. For example, suppose again that $R = 1.99$; the required wave-speed adjustment for $N = 2$ is minimal, equal to an adjustment of about 0.5% of the original wave speed when two reaches are used, whereas as we have shown the required interpolation $N = 1$ is quite large. One of the advantages of the hybrid approach presented next is that it can preserve the desirable flexibility of the rounding approach when using alternative interpolation strategies.

**FLEXIBLE DISCRETIZATION ALGORITHM**

The basic approach suggested here is to extend the characteristic back for an entire spatial step, as illustrated in Fig. 2 for the $C^+$ characteristic. The point at which this characteristic is exactly $\Delta t$ before the current point (i.e., point 1 in Fig. 2) determines the usual space line point, whereas the point exactly $\Delta x$ away (point 2) determines the usual time line point. However, because errors are associated with moving information from the grid points to the characteristic (i.e., interpolation), intermediate points such as point 3 in Fig. 2 may sometimes represent a better compromise.

Once this basic observation is made, the details of a more general approach quickly follow. If it is desired to interpolate

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**FIG. 1. Courant Number and Wave-Speed Adjustment as a Function of the Ideal Number of Reaches**
in both space and time, one need only determine the contribution or weight of the four nearest grid points surrounding the interpolated point in question. Borrowing the form of a linear basis from finite elements, a powerful interpolation function can be written as

$$\phi = \sum_{i=1}^{4} W_i \phi_i$$

(1)

where $\phi = \text{general interpolated variable (either head $H$ or discharge $Q$)}$, $\phi_i = \text{same variable at the adjacent grid points}$, and $W_i = \text{associated weight}$. If the nondimensional interpolation variable in the distance direction is $\xi$ and that in the time direction is $\eta$, then the equation can be written more explicitly for the $C^+$ characteristic leading to grid point $(n + 1, i)$ as

$$\phi = (1 - \xi)(1 - \eta)\phi_{i-1} + \xi(1 - \eta)\phi_i + (1 - \xi)\phi_{i+1} + \xi\phi_i$$

(2)

where the superscript $n = \text{temporal location}$; and the subscript $i = \text{spatial location of the associated grid points}$. Note that if no interpolation is required, $\xi = \eta = 0$, and thus $\phi = \phi_{i-1}$. Similar limiting values can be obtained at the other corners of the interpolated grid. Thus, for space-line interpolation $\xi = 0$ while $\eta = 0$ produces the appropriate weighting for time line interpolation. It is a simple matter to write the expressions for $\xi$ and $\eta$ in terms of Courant number for any selected interpolation procedure. Once the interpolated values of head and discharge have been obtained, the MOC solution can proceed as shown in many references (Wylie and Streeter 1993; Chaudhry 1987; Karney and McInnis 1992).

Other than the traditional space line and time line interpolation, a number of other interpolation procedures suggest themselves. Two logically appealing alternative interpolation approaches include minimum-point interpolation, which selects the point on the characteristic that lies closest to the primary MOC grid point $\phi_{i-1}$, and characteristic line interpolation, which interpolates in the direction of an intersecting characteristic. This final interpolation is particularly attractive because it joins the grid point to the required characteristic with a line segment that is itself a characteristic. Both of these approaches can be selected by simply choosing appropriate values of the corner weights. Moreover, once the basic procedure has been determined, further refinements are easily made. For example, wave-speed adjustments can be viewed as an offset of the Courant number before performing any interpolation, thus effectively rotating the characteristic curve to its adjusted slope prior to interpolation. After this, interpolation can take place using any reasonable strategy along the shifted characteristic. Such a hybrid approach allows a smooth and gradual transition between interpolation procedures and the wave-speed-adjustment approach.

Before presenting the details of the algorithm, one other approach to the discretization problem is worth mentioning. Rather than adjusting the wave speed, why not simply set the two interpolated parameters to zero, thus taking conditions at the foot of the characteristic to be equal to those at the nearby grid point? Thus, we may be tempted to simply set $\phi = \phi_{i-1}$ and not to calculate an adjusted wave speed. This wave-path-adjustment approach preserves the magnitude of the pressure wave caused by sudden flow stoppage but distorts the speed at which this information is transmitted to the remainder of the pipe system. However, numerical testing has indicated that in some systems instability can arise with this method. For this reason, the wave-path-adjustment approach is not recommended nor is it considered further here.

The general algorithm that combines all of these options into a single preprocessor step is written for compactness in standard C code and as shown in Fig. 3. This code requires specification of the pipe length ($L$), wave speed ($A$), and number of reaches in the shortest pipe (NRSP) where the shortest pipe has the minimum wave travel time $L/A$. The time step is set by the wave travel time for one reach in this shortest pipe, thus ensuring that all the pipes will have at least one reach. The algorithm is controlled using two primary parameters: INTERP, which selects the interpolation approach, and PSIMAX, a user-specified limit on the maximum allowable wave speed adjustment. As written, the parameter PSIMAX is used preferentially to avoid interpolation if possible and to preserve the desirable near-miss property of the round-off approach. If the nearest grid point is determined to be within striking distance of the allowable wave-speed adjustment, interpolation is avoided. If PSIMAX is set to zero, no wave-speed adjustment is allowed and the discretization weights are calculated using one of the pure interpolation approaches.

Fig. 4 illustrates the basic behavior of this hybrid approach. This graph displays the same data as Fig. 1, but now the pair of lines apply simultaneously to a single system. The solid line indicates the interpolation that remains after first using a maximum allowable wave-speed adjustment of 10%; the dashed line shows the required adjustment in wave speed. Note that with the 10% limit, no interpolation is required as long as the required number of reaches $R$ exceeds about 4.5. Moreover, the bound of either interpolation or wave-speed adjustment is considerably less than if either strategy is used exclusively.

The secondary parameter CRTL in the algorithm is used to set a time-line threshold for the courant number, thus illustrating some of the additional flexibility that can be incorporated into the hybrid approach. If the calculated Courant number in any pipe is close to 0.5, it is clear that the minimum interpolation approach should be based on time-line interpolation, because the foot of the (say) $C^+$ characteristic will pass near to the point $(i - 1, n - 1)$. Thus, by appropriately choosing CRTL, it is possible to force certain pipes to use time line interpolation, even if some other interpolation approach is being used for other pipes. In this case, it is logical to have the wave speeds of these pipes adjusted toward the previous time step (i.e., toward 0.5), with the switch criterion based on the Courant number. A value such as CRTL = 0.55 appears to work well in most circumstances. The extreme range of CRTL values is 0.50–1.0.

The attributes of the algorithm presented in Fig. 3 are summarized as follows:

- It can be easily applied to existing code
- The new algorithm is applicable to all pipes in a system, including those having only one reach
/* INTERP Interpolation Method */
0 = wavespeed adjustment
1 = spatial travel
2 = time-line
3 = 'minimum-point'
4 = 'characteristic line'
5 = 'wave path' adjustment

Step 1: Find minimum wave travel time */
R[i] = L[i]/A[i]; /* travel time = pipe length/wavespeed */
for (i=2; i<=WP; ++i)
if (R[i] = L[i]/A[i] < TMIN) TMIN = R[i];

/* Step 2: Find time step as wave travel time in 'shortest' pipe */
redisc:
DT = TMIN/WRS; /* WRS = number of reaches in 'shortest' pipe */

/* Step 3: Find weights for each grid point for each pipe */
for (i=1; i<=WP; ++i) {
R[i] = R[i] + DT; /* Exact (real) number of reaches */
W[i] = (R[i] - 0.50) + 1.0; /* Rounded (integer) number of reaches */
if (CR = W[i]/R[i] > 1.00/(1 - PSI)) CR = W[i]/R[i];
/* Shift CR (maximum allowable change in A controlled by PSIMAX) */
PSI = PSI MAX * CR; /* PSI = allowable shift in Courant Number CR */
if (fabs(1 - CR) <= PSI) CR = 1.0; /* If possible, avoid interpolation */
else if (INTERP == 0) {
/* Wavespeed change is too large */
WRS++; /* Increase number of reaches in shortest pipe */
goto redisc; /* Hence, try again with shorter time step */
else if (CR <= CRTL + PSI) CR = PSI; /* Closer to earlier step */
else CR = PSI; /* Increase CR as much as possible */
A[i] = CR * L[i] / (W[i] * DT); /* Obtain 'adjusted' wavespeed */
/* Obtain corner point weights based on CR and INTERP */
ZETA[i] = XI = 0.0; /* Default values for interpolation */
if (INTERP == 0 || INTERP == 5) {
/* Accept defaults */
else if (INTERP == 1 && CR > CRTL) ZETA[i] = 1 - CR; /* Space Line */
else if (INTERP == 3 && CR > CRTL) {
/* Minimum Point */
ZETA[i] = (1 - CR)/(1 + pow(CR, 2)); XI = CR * ZETA[i];
}
else if (INTERP == 4 && CR > CRTL) {
/* Characteristics Line */
ZETA[i] = 0.5*(1 - CR); XI = (1.0 - CR)/(2.0 + CR);}
else XI = (1.0 - CR)/CR; /* Hence, use Time Line interpolation */
W[i][0] = 1 - ZETA[i]; W[i][1] = (1 - ZETA[i]) * (1 - XI); W[i][2] = ZETA[i] * (1 - XI); /* Thus, knowing distances, ... */
W[i][3] = ZETA[i] * XI; /* ... find required weights. */
}

FIG. 3. C Code for Generalized Interpolation and Wave-Speed—Adjustment Algorithm

FIG. 4. Hybrid Discretization Combining Wave-Speed Adjustment and Interpolation

• It couples two traditional interpolation strategies and three new ones, with a controllable blend of the wave-speed—
  adjustment technique
• The procedure is a preprocessing step, so there is no run-
  time cost to the flexibility during the simulation of tran-
  sient events; all weights and adjustments are calculated
  only once for each pipeline in the system
• The term ZETA relating to the spatial interpolation is ob-
  tained during the discretization loop and can be stored to
  scale properly the integration of the friction term
• The use of wave-speed adjustment allows rounding, thus
  possibly reducing error in individual pipes and decreasing
  the overall error in wave travel time

Note that the demonstrated flexibility of the algorithm is not
exhaustive. For example, by adding an additional subscript it
is simple to index the interpolation approach to each pipe,
allowing different approaches to be used effortlessly in differ-
ent parts of the system. Again it is emphasized that there is
essentially no run-time cost to this flexibility. Furthermore, one
could easily improve the automatic discretization test in step
3 to include a number of other criteria, such as testing whether
the degree of interpolation is too large, even allowing for
wave-speed adjustments. For example, one could test whether
PSIMAX + (1 - C_i) ≤ TOL where TOL is some user-spec-
ified tolerance; if the test fails, the number of reaches could
be increased, exactly as the displayed algorithm does if the
required wave-speed adjustment is too large.

EHDE FOR FLEXIBLE ALGORITHM

An additional advantage of the flexible algorithm approach
developed here is that it allows simplification not only to the
numerical calculations, but also to its associated theoretical
analysis. Thus, the EHDE derivation shown in this section
combines several sets of equations presented in Ghidaoui and
Karney (1994) into a single collection of compact expressions.
These composite forms reduce to the previously presented
EHDEs for the special cases of pure space line interpolation and pure time line interpolation as well as for the method of wave-speed adjustment. In all cases, the template for comparison is the system of governing equations for water hammer in a pipe system.

If both the friction and the convective terms are negligible (as is typically assumed in error analysis), the characteristic form of the water hammer equations is (Wylie and Streeter 1993)

\[
\frac{dH}{dt} + \frac{a}{gA} \frac{dQ}{dt} = 0 \quad \text{if} \quad \frac{dx}{dt} = x = \pm a
\]  

(3)

where \( H = H(x, t) \) is piezometric head; \( Q = Q(x, t) \) is volumetric flow rate; \( A \) is cross-sectional area of the pipe; \( g \) is acceleration due to gravity; \( x \) is distance along the pipe’s centerline; \( t \) is time; and the ± notation is used to distinguish the positive and negative characteristics. As a result of the stated assumptions, these equations are nondissipative and nondispersive.

In essence, the EHDE for the flexible discretization algorithm is formulated by answering the question, "What is the system of equations that when integrated results in the same system of algebraic equations as are obtained by interpolating the fixed-grid MOC?" (Ghidaoui and Karney 1994). More specifically, the construction proceeds as follows. First, the hyperbolic-like terms are collected together. Second, the antiderivative concept is used to convert the algebraic expressions into integral equations. Third, because the latter equations are integrated with respect to both \( x \) and \( t \), a transformation of variables from \( x \) to \( t \) or vice versa must be invoked. The proper transformation is given by \( x(t) \) such that \( x \) equals the propagation speed of the numerical information. Finally, a limit argument shows that the integrand is zero. Carrying out these steps for the flexible algorithm produces the following set of equations:

\[
\frac{dH}{dt} \pm \frac{(1 + \psi)B}{1 - \psi} \frac{dQ}{dt} + \frac{\xi}{1 - \psi} \frac{d}{dt} \left[ \frac{dH}{dt} \pm (1 + \psi)B \frac{dQ}{dt} \right] = 0 \quad \text{if} \quad \frac{dx}{dt} = \pm \frac{a}{C},
\]

(4)

\[
\frac{dH}{dt} \pm \frac{(1 + \psi)B}{1 - \psi} \frac{dQ}{dt} + \frac{\xi}{1 - \psi} \frac{d}{dt} \left[ \frac{dH}{dt} \pm (1 + \psi)B \frac{dQ}{dt} \right] = 0 \quad \text{if} \quad \frac{dx}{dt} = \pm \frac{a}{2C},
\]

(5)

\[
Q^{+1} = (1 - \zeta)(\hat{Q}^{+1} + \zeta \hat{Q}^{+1}), \quad H^{+1} = (1 - \zeta)(\hat{H}^{+1} + \zeta \hat{H}^{+1}),
\]

(6)

where \( \psi \) is amount of wave-speed adjustment; \( \hat{Q}^{+1}, \hat{H}^{+1}, \hat{Q}^{+1}, \) and \( \hat{H}^{+1} \) are the solutions of (4) and that of (5), respectively; \( \zeta = [1 - C(1 + \psi)]/C(1 + \psi) \); and \( B = a/gA \).

Comparing (4)–(6) with (3) shows how the flexible discretization algorithm alters both the path and the magnitude of the water hammer waves. In particular, (4) and (5) show that the flexible discretization propagates flows and heads along the paths defined by \( \frac{dx}{dt} = \pm a/C \), not \( \frac{dx}{dt} = \pm a/2C \), and that is, paths that do not coincide with the true physical path except when \( C = 1 \) or \( 0.5 \) (i.e., \( \xi = 0 \) or \( \xi = 1 \)).

The EHDE shows that there are three different mechanisms responsible for artificially altering the magnitudes of \( Q \) and \( H \). First, the third term (involving the square brackets) in both (4) and (5) is a dissipative term when \( C < 1 \) and is an amplification term when \( C > 1 \). As amplification produces instability, then only \( C \approx 1 \) is acceptable, thus confirming that the Courant condition is required to ensure the third term is dissipative. This flux of momentum term is one way by which the flexible discretization algorithm dissipates the peak of the waves. Second, \( H \) and \( Q \) as defined by (6) are weighted averages of two waves: one traveling along \( \frac{dx}{dt} = a/C \), while the other travels along \( \frac{dx}{dt} = a/2C \); thus, the wave peak of \( H \) and \( Q \) is always lower than the maximum of the two wave peaks of these two waves (Ghidaoui and Karney 1994). Third, the magnitude of \( H \) and \( Q \) also is influenced by the fact that the multiplier of \( dQ/dt \) in (4) and (5) is \( (1 + \psi)B \) and not \( B \). Whether this wave speed produces a wave attenuation or a wave amplification depends on the wave travel time and the boundary conditions of the problem (Karney and Ruus 1985). It is clear from the EHDE that all three errors affecting the magnitude of \( H \) and \( Q \) are dependent on \( C \) and \( \psi \). Hence, for a given \( C \), (i.e., level of discretization) the analyst is able to change \( \psi \) so as to try and minimize the sum of the three errors. For example, setting \( \psi = \xi/1 - \xi \) reduces the error associated with the momentum flux to zero and changes the error associated with superposition as well as the error associated with wave-speed adjustment. Hence, it is clear that the total error is a function of the three individual errors, which are in turn a function of \( \psi \) for a given discretization. The minimum path interpolation is an attempt to minimize that part of the total error, which strictly depends on the geometry. However, this is not sufficient to ensure that the total error is minimized as this error also depends on the wave profile.

The flexible discretization scheme is consistent because a \( C \), of either 1 or 0.5 reduces the system (4)–(6) to the original system of equations defined by (3). In addition, the flexible algorithm is stable for \( C \leq 1 \), because this is the only way to ensure both that the numerical wave speed is larger than the physical wave speed and that the error coefficients are consistent with the required dissipation.

**EXAMPLE APPLICATION**

Given space constraints, it is impossible to test systematically the flexible algorithm for the desirable number of realistic pipeline systems and transient events required to draw general conclusions. Rather than attempting to be definitive, we test a relatively simple series pipeline system subject to a linear valve closure and draw primarily anecdotal conclusions from the observed numerical results. However, unlike some of the test cases reported in the literature, such as those performed on a single pipe of uniform properties, the discretization problem in the example is genuine—the three pipe segments have different lengths and different wave speeds. The imposed variation in property values is sufficient to create not only a discretization problem, but also a challenging set of wave reflections and transmissions. To ensure that the numerical tests are realistic, as well as to emphasize the differences between the interpolation approaches, all the numerical tests are run with a small number of reaches, because it is only those pipes that truly present a discretization problem. Moreover, we have resisted the temptation to create a test system displaying near-miss behavior that gives such a decided advantage to the wave-speed—adjustment approach. Thus, the reported behavior accurately reflects the difficulties that can be associated with the discretization problem.

The simulated system consists of three series-connected pipe segments with lengths and wave speeds as shown in Table 1. The pipes are numbered sequentially from an upstream reservoir with a constant head of 200 m to a downstream control valve. The pipeline carries an initial steady-state discharge of 1.0 m³/s. Each pipe has a constant Darcy-Weisbach friction factor \( f \) of 0.015 and each is 1.0 m² in cross-sectional area; no losses are assumed at the pipe junctions. The travel time for a wave to pass from one end of the system and back again
(2L/a for the system as a whole) is 1 s. The effective area of the downstream valve is assumed to decrease linearly to zero in 1.5 s (or 3L/a), thus giving sufficient time to allow interaction between the valve closure and the upstream boundary condition. The wave-propagation characteristics of the pipe segments are chosen so that if NRSP is set to any integer multiple of four, the Courant condition is exactly satisfied in each pipe. Thus, the standard for numerical comparison, plotted in each graph in Figs. 5–7, is an accurate simulation based on using eight reaches in the hydraulically shortest pipe (i.e.,

<table>
<thead>
<tr>
<th>Pipe</th>
<th>Length L (m)</th>
<th>Wave speed a (m/s)</th>
<th>L/a (s)</th>
<th>Number of Reaches R</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>110</td>
<td>1.100</td>
<td>0.100</td>
<td>1, 2, 8</td>
</tr>
<tr>
<td>2</td>
<td>175</td>
<td>1.000</td>
<td>0.175</td>
<td>1.75, 3.5, 14</td>
</tr>
<tr>
<td>3</td>
<td>270</td>
<td>1.200</td>
<td>0.225</td>
<td>2.25, 4.5, 18</td>
</tr>
</tbody>
</table>

NRSP = 8). This choice not only eliminates the need either to interpolate or to adjust the wave speed, but ensures that other numerical errors, such as the integration of the friction term and distortions caused by the complex pattern of reflections at pipe junctions, are small.

As Table 1 shows, the pipe with the minimum wave travel time (min L/a) in this system is the first one. The time step for the simulation is set by dividing the minimum wave travel time by NRSP (i.e., steps 1 and 2 in the algorithm). Thus, once R has been selected and the time step is determined, the exact (noninteger) number of reaches in each of the other two pipes can be calculated. Note that the R values shown in Table 1 are the only discretization information that is relevant: once the number of reaches is set, the associated Courant number or required wave-speed adjustment, or some combination of these two approaches, is fixed. This mapping between the ideal number of reaches and the discretization is explicit in the algorithm (Fig. 3) and its graphical form is shown in Figs. 1 and 4. The number of reaches associated with two coarse discretizations (NRSP = 1 or 2) are shown in Table 1 as well as those for the noninterpolation or exact case having NRSP = R = 8. The time step thus varies from a maximum of 0.10 s to a minimum of 0.0125 s. Table 1 also makes it clear that the discretization error in this system is significant when NRSP is small.

RESULTS

Some representative results from simulating the example system with the flexible algorithm are summarized in Figs. 5–8. In each case, the plots show the variation of head at the valve end as a function of time. The graphs contain three sets of curves, one of which is always a heavier solid line representing the accurate (nominal) exact eight reach case. Fig. 5 shows the results when using NRSP = 1 and 2 for space line interpolation and Fig. 6 shows the comparable results for time line interpolation. Fig. 7 repeats this pattern for the characteristic line interpolation as does Fig. 8 for the minimum-point interpolation.

Several observations are evident from the displayed results. In all cases, the more rapidly decaying curve (dashed) relates to the coarse discretization and the intermediate curve is associated with the two reach results. Thus, the numerical attenuation of the wave is dramatic when the discretization is poor. Moreover, all the methods perform more accurately with a
smaller time step, as indicated by the significant improvement in both dissipation and the timing and shape of pressure peaks between the two reach and one reach results. In fact, the least accurate results when using NRSP = 2 with any of the four methods are better than the most accurate results when using NRSP = 1. Clearly, reducing the time step is a much better way of improving accuracy than changing the interpolation approach. However, it should be pointed out again that the improvement in the results with smaller time steps is by no means monotonic. For example, in the current example if NRSP is set to 5, 6, or 7, the accuracy of the results is actually worse than if NRSP = 4.

One of the intriguing aspects of the plotted results is that, at least for the coarse discretization, time line interpolation actually increases the predicted pressure associated with the first peak (Fig. 6). How is this possible? The key to understanding this response is to realize that interpolation strategies do not move information in a physical fashion. In this example of a nonsudden closure, wave reflections from the reservoir help to decrease the pressure rise at the valve end. However, with time line interpolation and a small number of reaches, the returning wave is attenuated heavily. Thus, paradoxically, the severity of the attenuation caused by time line interpolation means that too little of the wave generated during the early stages of valve closure is reflected back from the upstream reservoir; hence, too little of this wave is present to reduce the head rise associated with the latter stages of valve closure. The time line result is similar to what would be observed if the valve were to close more rapidly than the assumed 1.5 s. Interestingly, if the valve closure had been assumed to occur more rapidly, say in 1 s or less, this overestimation is not observed. In fact, it is easy to show that for a 1-s valve closure, time line interpolation produces more accurate predictions of the first pressure peak than any of the other interpolation techniques presented here. These observations confirm the contention of Ghidaoui and Karney (1994) that sudden closure may not always be the most conclusive test for interpolation studies.

One of the insights that the EHDE approach supplies is that the performance of an interpolation procedure is influenced significantly by the form of the wave profile. This raises an important question that frequently is overlooked in interpolation studies. Are head H and discharge Q profiles equivalent in the space and time directions? In fact, our experience suggests that this equivalence seldom exists. Disturbances in pipe-systems are initiated generally at the boundaries and propagate through the pipeline system until they reflect or interact with other boundaries. As a result, the variation of head or discharge with time at a given point is quite different than the variation of H and Q with distance at a given time. Thus, interpolation of spatial information often gives quite different results than interpolation of temporal information. This fact makes generalizations about which interpolation approach to recommend difficult, but it also makes the flexible interpolation algorithm particularly valuable, because the choice of approach can so easily be tailored to the specific problem.

Generally, these observations are supported by the current study. For the one reach case, although all interpolation approaches performed poorly, there are important differences. The dramatic dissipation associated with time line interpolation is somewhat more gradual when using space line interpolation (see Figs. 5 and 6). The decay associated with minimum-point and characteristic line interpolation is quite similar to the space line case, though the timing of the peak is more accurately maintained with the hybrid approach. In fact, our experience indicates that characteristic line interpolation often accurately preserves timing information while doing a reasonable job in controlling the dissipation.

Finally, Fig. 9 compares conventional time line interpolation with a hybrid approach based on time line interpolation with an 8% maximum wave-speed adjustment. Both results are computed for a coarse discretization using only one reach in the shortest pipe. Notice in Fig. 9 that the peak magnitude comes into better agreement with the accurate eight-reach solution when the wave-speed adjustment is added to the time line interpolation. This observation is consistent with earlier comments made when deriving the EHDEs: the best procedure must balance the properties of the interpolation approach with the behavior of the pipe system and its boundaries.

CONCLUSIONS

When an analyst is introduced to the discretization problem of selecting a time step in pipeline applications, it gives the misleading appearance of simplicity. This appearance has almost certainly contributed to the large variety of discretization approaches that have been proposed. If the physical parameters of the pipe system as well as the values of the dependent variables at the adjacent nodes are known, it appears to be a simple matter to produce a good estimate of the same variables...
at the foot of the characteristic. In reality, however, a good estimate, using any interpolation procedure, is difficult to obtain in hyperbolic problems. The basic problem is this: because derivatives are often indeterminate across characteristics, a Taylor series cannot accurately span characteristic curves. Thus, even if a function and the value of all its derivatives are known at a point in a wave-propagation problem, function values at surrounding points can seldom be determined with certainty. This means that, fundamentally, the interpolation problem cannot be resolved. Of course, for the analyst this recognition does not eliminate the problem; it is important to do as well as possible with somewhat limited and faulty procedures. Because it is highly unlikely that any one method will be best for all circumstances, this paper concentrates on practical, efficient, and flexible approaches that can easily be altered as the need arises.

More specifically, the flexible discretization procedure presented in this paper is computationally efficient because, at least for constant wave speeds, it can be configured as a preprocessing step with little run-time cost. Moreover, the method is flexible, because a variety of interpolation techniques, as well as blends of these approaches with the wave-speed adjustment technique, can be readily selected. Thus, adjustment between various methods is smooth and continuous, allowing different combinations to be selected for different applications without either having to change the time step or to recode.

Both the formal EHDE approach and experience indicates that all discretization errors decrease as the Courant number (or the required wave-speed adjustment) in each pipe approaches unity. This effect is strongly evident in the presented results. For all interpolation approaches the larger number of reaches dramatically improves the accuracy of the solution. For this example, even the worst two-reach interpolation approach (time line) performs as well as the best approach with a single reach (space line). Moreover, some degree of wave-speed adjustment, even when blended with a poor interpolation approach for a given problem, can be very beneficial in controlling numerical attenuation. The interesting moral seems to be that it is sometimes preferable to simulate accurately an approximate system than to simulate inaccurately the correct system.

For many pipeline systems, interpolation errors may be significant if a large time step is used. In fact, it is possible that these errors may outweigh problems due to the number of nonlinear terms such as friction. The flexible interpolation algorithm presented here should serve as a useful tool for numerically investigating these interpolation errors, although reducing the time step is often a safer and more successful strategy than adjusting the method of interpolation for a given time step. It must be emphasized that all interpolation strategies, including the flexible approaches reported in this paper, perform poorly in some systems when the degree of interpolation is high. Because of the nature of the hyperbolic equations, small time steps appear essential for minimizing interpolation errors. Thus, if accuracy is important, or when in doubt, the best advice appears to be to decrease the time step. Both the numerical examples and the EHDE approach presented in this paper show that interpolation approaches fundamentally change the physical problem and must be viewed as a nontrivial transformation of the governing equations.

APPENDIX I. REFERENCES

APPENDIX II. NOTATION
The following symbols are used in this paper:

\[
\begin{align*}
A &= \text{cross-sectional area of pipe;} \\
\alpha &= \text{wave celerity;} \\
B &= \text{pipe constant } a/a; \\
C &= \text{Courant number;} \\
D &= \text{inside pipe diameter;} \\
f &= \text{Darcy-Weisbach friction factor;} \\
g &= \text{acceleration due to gravity;} \\
H &= \text{piezometric head; } H = H(x, t); \\
H_1, H_2 &= \text{interpolated piezometric head; } (h), (\bar{h}) = \text{piezometric head corresponding to } a/C, \text{ and } a/2C; \\
i &= \text{integer denoting spatial nodal location; } \\
l &= \text{integer denoting time nodal location; } \\
Q &= \text{volumetric flow function; } Q = Q(x, t); \\
Q_i &= \text{volumetric flow at node } (n, i); \\
Q_1, Q_2 &= \text{volumetric flow corresponding to } a/C, \text{ and } a/2C; \\
t &= \text{time, time step; } \\
x, \Delta x &= \text{space coordinate, space increment; } \\
x &= \text{speed of propagation, } dx/dt; \\
\psi &= \text{wave-speed adjustment; } \\
\xi &= \text{degree of space line interpolation; } \\
\zeta &= \text{degree of time line interpolation.}
\end{align*}
\]