MODIFIED TRANSFORMATION AND INTEGRATION OF 1D WAVE EQUATIONS

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ABSTRACT: This technical note introduces an alternative method of transforming hyperbolic partial-differential equations into characteristic form. The method is based on transforming the governing equations to a reference frame moving with finite speed $u$. Thus, the method is analogous to the "moving observers" used traditionally in graphical water-hammer theory to solve the equations of motion [e.g., Parmakian (1963) and Bergeron (1961)] or to the method of deriving simplified governing equations by using a translating reference frame [e.g., Henderson (1966)]. The difference in the present case is that although the governing equations are assumed to be known, they are transformed into characteristic form by a shift in reference frame. In essence, the transformation uses the total derivative concept and is both simple and insightful. In fact, for both open-channel flow and water-hammer applications, it is shown that by transforming only the continuity equation along a characteristic curve, the dynamic equation naturally arises during the transformation. A mathematical justification and generalization of the proposed method is provided.

ALTERNATIVE METHOD OF TRANSFORMING WAVE EQUATIONS

There are two common approaches in the literature for transforming the governing equations for open-channel or closed-conduit transient flow into characteristic form. These methods are the $\lambda$ approach [e.g., Lister (1960); Wylie and Streeter (1993); and Chaudhry (1987)] and the eigenvalue method [e.g., Abbott (1986)]. The first approach linearly combines the partial differential equations representing the continuity and momentum relations and then solves for values of $\lambda$ that transform this set of equations into ordinary differential equations. The eigenvalue method accomplishes the same goal by deriving the characteristic equations from the eigenvalues of the convection matrix. Neither method provides the flexibility of transforming the equations along an arbitrary curve. Yet having this flexibility provides a way of constructing new numerical schemes and gives theoretical error estimates of various numerical procedures [see Ghidaouie and Karney (1994)]. The alternative method presented here is still based on the total derivative concept, but has the attributes of being simple and physical. The water-hammer and the Saint-Venant equations illustrate how the method is applied. Finally, the method is justified both physically and mathematically.

Water-Hammer Equations

The pressure fluctuations created by a flow disturbance in a closed conduit propagate as pulse waves. If it is assumed that the fluid is compressible, the pipe is elastic, the convective terms $v(\partial Q/\partial x)$ and $v(\partial Q/\partial t)$ and the slope term are negligible, the friction force is modeled by the steady-state Darcy-Weisbach equation, the flow is one-dimensional, and the pipe has a circular cross section, then the resulting momentum and continuity equations are as follows (Wylie and Streeter 1993; Chaudhry 1987):

\[
\frac{dQ}{dt} + gA \frac{dH}{dP} + RQ|Q| = 0
\]  

\[
c^2 \frac{dQ}{dx} + gA \frac{dH}{dt} = 0
\]

in which $t =$ time; $x =$ distance along pipe centerline; $H = H(x, t) =$ piezometric head; $Q = Q(x, t) =$ volumetric rate of flow or discharge; $g =$ acceleration due to gravity; $A =$ cross-sectional area of a pipe with diameter $D$; $f =$ Darcy-Weisbach friction factor; $c =$ wave speed; and $R = f/2DA$; a constant. For convenience, partial derivatives of variables with respect to distance or time are written as subscripted terms.

If $H = H(x, t)$ and $Q = Q(x, t)$, then the total derivatives of head and flow with respect to time are as follows:

\[
\frac{dH}{dt} = H_t + \frac{dx}{dt} H_x
\]

\[
\frac{dQ}{dt} = Q_t + \frac{dx}{dt} Q_x
\]

If the speed of a moving observer is

\[
\frac{dx}{dt} = u
\]

then the total derivatives represent the variation of head and discharge along the observer's path.

If the observer's speed $u$ is nonzero, the total derivative expressions can be solved for $H_t$ and $Q_t$, and the result substituted into (2), thus producing the following continuity equation transformed to the observer's path:

\[
\frac{c^2}{u} \left( \frac{dQ}{dt} - Q_t \right) + gA \left( \frac{dH}{dt} - uH_t \right) = 0
\]

Dividing this result by $u$ and rearranging yields

\[
\left( \frac{c}{u} \right)^2 \frac{dQ}{dt} + gA \frac{dH}{dt} = \left[ \left( \frac{c}{u} \right)^2 - gA \right] Q_t + gAH_t = 0
\]

which is the general form of the transformed continuity equation. However, in the special case that the observer moves with the wave speed ($u = c$), then (7) reduces to

\[
\frac{dQ}{dt} + gA \frac{dH}{c} = 0
\]

Alternatively, when $u = -c$, then (7) gives

\[
\frac{dQ}{dt} - gA \frac{dH}{c} = 0
\]

As shown in Fig. 1, (8) and (9) represent the C+ and C− compatibility relations, respectively. In other words, these equations depict the mass-conservation principle as seen by
two imaginary observers, one moving with a velocity $c$ and the other $-c$. It is clear that the continuity equation alone cannot give the complete description of the transient wave along the characteristic lines because partial derivative terms still exist in (8) and (9). In fact, these partial derivative terms are easily identified from the momentum equation (1) as the friction term. Using this identification converts (8) and (9) into the full characteristic expressions for water hammer

$$
\frac{dQ}{dt} + \frac{gA}{c} \frac{dH}{dt} + RQ|Q| = 0 \tag{10}
$$

in which the positive sign relates to the $C^+$ characteristic and the negative sign refers to the $C^-$ characteristic.

To summarize, this transformation procedure shows that although the continuity equation can be written along any path $u$, it is only along the wave path ($u = \pm c$) that the partial derivative terms can be eliminated using the momentum equation. If the mathematical and the wave path do not coincide, the observer’s view of the problem is distorted and the partial derivative terms continue to exist. Moreover, it is easy to show that is is not necessary to begin the transformation with the continuity equation; the full characteristic equations can also be derived by starting with a transformed momentum relation.

**Open-Channel Transient Flow Equations**

The Saint-Venant equations, frequently used to model transient flow in open channels, can be written as follows [e.g., Wylie and Streeter (1993)]:

$$
v_i + vv_i + gy_i + g(S_f - S_o) + (q/A)v = 0 \tag{11}
$$

$$
Ty_i + vv_T + Av_i = 0 \tag{12}
$$

in which $t$ = time; $x$ = distance along the channel; $v(x, t)$ = average velocity; $y(x, t)$ = depth of the fluid; $A(x, t)$ = wetted cross-sectional area of the channel; $T(x, y)$ = top width of $A(x, t)$; $S_f$ = slope of the energy grade line; $S_o$ = slope of the bed of the channel; $q$ = lateral inflow per unit length of the channel; $g$ = acceleration of gravity; and partial derivatives are again written as subscripted variables.

Although the basis of the transformation method for the equations of open-channel flow is the same as for the water-hammer case, the mathematical derivation is a little more involved. This is because both the original equations and their characteristic form are more complex. However, the generalized mathematical approach presented in a subsequent section shows that the specifics of the transformation are in fact tightly constrained, despite the seemingly arbitrary steps that follow.

If the total derivative expressions for $y$ and $v$ are written and then solved for $y$, and $v$, respectively, the results can be substituted into (12) and rearranged to give

$$
T[1 - (v/A)[dy/dt] + (Tv/A)y, + uy,] - Tuy,
+ vv_Ty, + (A/\mu)[dv/dt] - v, - q = 0 \tag{13}
$$

Dividing (13) by $A/\mu$ and rearranging produces

$$
(dv/dt) + (T/A)(u - v)(dv/dt) + (\mu/A)(Ty, + Tv, - q)
- (u - \mu^2)(T/A)y, - v, - (q/A)(u - v) = 0 \tag{14}
$$

From (12) the term $v(Ty, + Tv, - q)/A$ is equal to $-vv,$. Substituting this result into (14) and rearranging produces

$$
(dv/dt) + (T/A)(u - v)(dv/dt) - [v, + vy, + gy,]
+ (u - \mu^2)(T/A)y, + (q/A) - (q/A)(u - 2v) = 0 \tag{15}
$$

Eq. (15) is the continuity equation transformed along a general path. For the special case when this path is given $u = v + c$, where $c = \sqrt{gA/T}$, the continuity equation becomes

$$
(dv/dt) + (g/c)(dv/dt) + [v, + vy, + gy,]
+ (q/A) - (q/A) - (v - c) = 0 \tag{16}
$$

Now, the third term in (16) can be easily evaluated from the momentum equation (11) as $g(S_f - S_o)$. Inserting this result and rearranging reduces the above expression to the following form

$$
(dv/dt) + (g/c)(dv/dt) + g(S_f - S_o) + (q/A)(v - c) = 0 \tag{17}
$$

which is the $C^-$ compatibility equation for open-channel transient flow. Similarly, when the path of transformation is defined by $u = v - c$, then by coupling (15), (11), and using the fact that $c^2 = ga/T$ produces the $C^-$ compatibility equation for open-channel transient flow

$$
(dv/dt) - (g/c)(dv/dt) + g(S_f - S_o) + (q/A)(v + c) = 0 \tag{18}
$$

Thus, as in the water-hammer case, the transformation from partial differential equations into ordinary differential equations can be made by writing the continuity relation along the characteristic lines using the concept of total derivative. In fact, it is only along the characteristic lines that the momentum equation naturally arises.

**Mathematical Justification**

It is quite straightforward to extend the method presented in the previous sections to a general 1D system of quasilinear partial differential equations. This general derivation also illustrates the way the transformation method presented here compares with the traditional eigenvalue approach.

Let a 1D system of quasilinear partial differential equations be written as follows:

$$
(\partial u/\partial t) + B(t, x, u)(\partial u/\partial x) = f(t, x, u) \tag{19}
$$

in which $u = (u_1, u_2, \ldots, u_n)'$ = unknown vector of dependent variables; $f = (f_1, f_2, \ldots, f_n)'$ = column vector that represents sources, sinks, dissipation, and so on; $B$ = convective square matrix of dimension $n$; $t =$ time; and $x =$ distance. If $B$ has $n$ linearly independent eigenvectors, then $B$ is diagonalizable. That is, there must exist an invertible matrix $P$ and a diagonal matrix $A$ such that $B = P^{-1}AP$, where $P^{-1}$ = inverse of $P$. Hence, (19) can be rewritten as follows:

$$
P(\partial u/\partial t) + AP(\partial u/\partial x) = Pf \tag{20}
$$

It is always possible to find a variable transformation from $u$ to $w$ such that (20) can be written as follows:

The total derivative of \( w \) is given by
\[
\frac{dw}{dt} = \frac{\partial w}{\partial t} + \mathbf{A} \frac{\partial w}{\partial \mathbf{x}} = \mathbf{P} \frac{\partial w}{\partial \mathbf{x}}
\]  
(21)

where \( \mathbf{I} \) = identity matrix; and \( \mathbf{dx}/dt = \) vector defining the path of transformation of \( \mathbf{u} \). Multiplying (22) by \( \mathbf{P} \) and rearranging produces
\[
\mathbf{P} \frac{dw}{dt} = \mathbf{P} \frac{\partial w}{\partial t} + \mathbf{A} \frac{\partial w}{\partial \mathbf{x}} + \{ \mathbf{I} \frac{d\mathbf{x}}{dt} - \mathbf{A} \} \mathbf{P} \frac{\partial w}{\partial \mathbf{x}}
\]  
(23)

Using (21) to evaluate the first two terms on the right-hand side of (23) gives
\[
\mathbf{P} \frac{dw}{dt} = \mathbf{P} \frac{\partial w}{\partial t} + \mathbf{A} \frac{\partial w}{\partial \mathbf{x}} + \{ \mathbf{I} \frac{d\mathbf{x}}{dt} - \mathbf{A} \} \mathbf{P} \frac{\partial w}{\partial \mathbf{x}}
\]  
(24)

which is the transformed form of the quasilinear system of hyperbolic differential equation along an arbitrary path defined by \( \mathbf{dx}/dt \). It is clear from (24) that only if \( \mathbf{dx}/dt = \mathbf{A} \) are the additional partial derivative terms eliminated. That is, only when the path of transformation coincides with the characteristic path that one does not need to "invent" terms to correct for the distorted view of the wave.

**CONCLUSION**

The equations governing 1D unsteady flows can be transformed into ordinary differential equations using the total derivative concept. This new method is both simple and physical. For example, if the continuity equation is written along the characteristic lines and the total derivative concept is applied, then the characteristic equations arise naturally during the transformation. Thus, for both the open-channel flow and the water-hammer equations, the transformation reveals how the continuity and momentum relations are interrelated by the characteristic method. In addition, it is shown that the method is applicable to systems of partial differential equations, thus making connections with the traditional eigenvalue method of transforming hyperbolic equations into characteristic form.

This new transformation procedure shows that although the continuity equation can be written along any path \( \mathbf{dx}/dt = \mathbf{u} \), it is only along the wave path that the partial derivative terms can be identified using the momentum equation. If the mathematical and the wave path do not coincide, the observer's view of the problem is distorted and the partial derivative terms will continue to exist. Moreover, it is not necessary to begin the transformation with the continuity equation; the full characteristic equations can also be derived starting from the momentum relation.

Both physical examples show that those partial derivative terms that continue to exist along the characteristic lines account for the frictional losses in the water-hammer case and for both friction and lateral inflow in the open-channel problem. This highlights the fact that only path-independent physical quantities can be written as total derivatives. Only if there are no losses and no sources and/or sinks in the mathematical model are the additional partial derivative terms zero along the wave path; however, along any path other than the characteristic, the partial derivative terms continue to exist.

Numerically, the persistence of additional terms along any path other than the characteristic helps explain why not only the order of approximating the governing equations, but also the path of propagation of information, is important for numerical accuracy. If the numerical wave path does not coincide with the characteristic path, then additional terms are required to account for the distorted view of the problem. The additional terms are usually first-order derivatives, which determine the overall order of a numerical scheme—a crucial insight that is gained through the flexible transformation method. In addition, the fact that the proposed method allows the analyst to choose the path of transformation at will can be used to eliminate the discretization problem by choosing \( \mathbf{u} \) so that \( C = 1 \). The additional partial derivative terms can then be approximated using a finite-difference procedure. Moreover, transforming the governing equation along the numerical wave path and comparing the result to the equivalent hyperbolic differential equation provides a theoretical estimate of the numerical errors (Ghidaoui and Karney 1994).

**APPENDIX I. REFERENCES**


**APPENDIX II. NOTATION**

The following symbols are used in this paper:

- \( A \) = cross-sectional area of pipe and/or channel;
- \( B \) = convective square matrix of dimension \( n \);
- \( f \) = Darcy-Weisbach friction factor;
- \( \mathbf{f} \) = vector of known quantities;
- \( H \) = piezometric head; \( H = H(x, t) \);
- \( q \) = acceleration due to gravity;
- \( \mathbf{I} \) = identity matrix;
- \( n \) = integer denoting time nodal location;
- \( \mathbf{P} \) = invertible matrix of eigenvectors;
- \( Q \) = volumetric flow function: \( Q = Q(x, t) \);
- \( u \) = convective velocity;
- \( \mathbf{u} \) = vector of dependent variables;
- \( v \) = average velocity; \( v = v(x, t) \);
- \( x \) = space coordinate;
- \( \Delta t \) = time step;
- \( \mathbf{A} \) = diagonal matrix of eigenvalues; and
- \( \lambda \) = multiplier in characteristics method.