

# EQUIVALENT DIFFERENTIAL EQUATIONS IN FIXED-GRID CHARACTERISTICS METHOD

By Mohamed S. Ghidaoui<sup>1</sup> and Bryan W. Karney<sup>2</sup>

**ABSTRACT:** Various numerical techniques, such as wave-speed adjustment or interpolation, are generally required in order to apply the fixed-grid method of characteristics to multipipe systems or to systems with variable wave speed. However, these techniques introduce into the solution unwanted side effects such as numerical attenuation and dispersion. The present paper develops the concept of an equivalent hyperbolic differential equation to study how discretization errors arise in pipeline applications for the most common interpolation techniques. In particular, it is shown that space-line interpolation and the Holly-Preissmann scheme are equivalent to a wave-diffusion model with an adjusted wave speed, but that the latter method has additional source and sink terms. Further, time-line interpolation is shown to be equivalent to a superposition of two waves with different wave speeds. In general, the equivalent hyperbolic differential equation concept evaluates the consistency of the numerical scheme, provides a mathematical description of the numerical dissipation and dispersion, gives an independent way of determining the Courant condition, allows the comparison of alternative approaches, finds the wave path, and explains why higher-order methods should usually be avoided.

## INTRODUCTION

Transient flow in closed conduits is governed by the quasi-linear momentum and continuity equations in the space-time ( $x, t$ ) plane. The method of characteristics (MOC) is often used to transform the governing partial differential equations into a system of four ordinary differential equations—that is, two compatibility equations that are valid along two sets of characteristic lines. However, the transformation of the domain from the general ( $x, t$ ) plane to a set of characteristic lines becomes problematic when the fixed-grid MOC is applied to multipipe networks or to systems with variable wave speed. These problems occur because the fixed-grid MOC ideally requires all pipes to satisfy the Courant condition exactly. Yet the pipes in a system tend to have different wave speeds and various lengths; hence, the constraint of a common  $\Delta t$  makes it extremely unlikely that there will be an integer number of reaches in all conduits. Moreover, the wave speed can vary in a pipe due to phenomena such as air release, thus making it impossible to satisfy the Courant condition exactly for all time steps. To circumvent these complications, the analyst can either interpolate between grid points or adjust the wave speed. The former approach results in numerical dissipation and dispersion, whereas wave-speed adjustment, although nondissipative and nondispersive, distorts the physical characteristics of the problem.

---

<sup>1</sup>Lect., Dept. of Civ. Engrg., Hong Kong Univ. of Sci. and Technol., Hong Kong; formerly, Grad. Student, Dept. of Civ. Engrg., Univ. of Toronto, Toronto, M5S 1A4, Canada.

<sup>2</sup>Assoc. Prof., Dept. of Civ. Engrg., Univ. of Toronto, Toronto, M5S 1A4, Canada.

Note. Discussion open until March 1, 1995. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on February 19, 1993. This paper is part of the *Journal of Hydraulic Engineering*, Vol. 120, No. 10, October, 1994. ©ASCE, ISSN 0733-9429/94/0010-1159/\$2.00 + \$.25 per page. Paper No. 5623.

The most widely used technique for analyzing the numerical errors associated with interpolation techniques is the Fourier method (Wiggert and Sundquist 1977; Goldberg and Wylie 1982; Katopodes 1984; Yang and Hsu 1990). The Fourier approach ignores the boundary conditions and assumes that the initial conditions are periodic (O'Brian et al. 1951; Dammuller et al. 1989; Samuels and Skeels 1990; Sibetheros et al. 1991). As a result of these and other shortcomings, alternative approaches have been proposed. For example, Chaudhry and Hussaini (1985) employed  $L_1$  and  $L_2$  norms, and Sibetheros et al. (1991) used three dimensionless parameters to study various numerical errors. However, these latter two methods are only applicable to problems in which the exact solution is known. More general techniques include the mass balance principle used by Sivaloganathan (1978) and Wylie (1980) to study the numerical errors in open channel flow and the energy balance approach used by Ghidaoui and Karney (1992) to study interpolation errors in water-hammer applications.

This previous research has generally concentrated on the quantification of discretization errors. The goal of the present paper is to develop a theoretical framework for the analysis and classification of the numerical errors associated with various interpolation techniques as well as the wave-speed adjustment approach. This framework is based on deriving the system of differential equations that result in the same algebraic expressions as those obtained by the MOC solution with interpolation. This system of equations is given the name equivalent hyperbolic differential equations (EHDE) to emphasize its fundamental differences with conventional equivalent differential equations (EDE). However, before developing and applying the EHDE concept, a brief review of the conventional EDE approach is presented.

### Equivalent Differential Equations

Time-dependent partial differential equations are frequently integrated using finite-difference approximations. The difference equation is developed by replacing the partial derivatives in the equation by their Taylor series expansion around a nodal value. Ideally, if the vector of dependent variables is analytic, then the nontruncated difference equation is analogous to the partial differential equation. To solve the untruncated difference equation, the value of each dependent variable and all its derivatives must be obtained at every node. Of course, this is not possible. Thus, the infinite difference equation must be truncated. As a result, the solution of the partial differential equation and that of the difference equation are not identical.

To understand the nature of the numerical errors one must analyze the properties of the difference equation. This analysis can be best performed by constructing the EDE of the difference scheme. The traditional approach for constructing EDEs is to replace the terms in the difference equation by their Taylor expansions (O'Brian et al. 1951; Yanenko et al. 1983; Leonard 1979; Noye 1991). The associated solution of the EDE is a continuous function that is equal to the solution of the difference equation at all grid nodes. The numerical errors introduced into the mathematical model by the finite-difference solution can be analyzed theoretically by comparing the governing system of partial differential equations to the system of equivalent partial differential equations arising from the difference equation. In general, the partial differential operator of the EDE consist of the terms in the partial differential operator of the governing equation in addition to an infinite series of nonphysical partial differential terms. These extra terms can be used to study the properties of the numerical scheme. For example,

if these extra terms tend to zero as the grid size tends to zero, then the finite-difference equation is consistent with the physical model. Further, if the governing equation is used to transform the nonphysical temporal derivatives into spatial derivatives, the presence of even order derivatives implies that the numerical scheme is dissipative, whereas odd order derivatives imply that the scheme is dispersive (Yanenko et al. 1983; Leonard 1979; Noye 1991).

It must be emphasized, however, that the construction of the finite difference and its associated EDEs presupposes that the vector of dependent variables is analytic everywhere in the physical domain. This is a highly questionable assumption in hyperbolic problems where sharp discontinuities in the initial and boundary conditions are propagated to internal nodes. In fact, the vector of the dependent variables is piecewise analytic only along the characteristic curves. Thus, the generic analysis based on the order of the truncation error is not conclusive and one should be cautious even to use the term "higher order" for this kind of equation. Moreover, Noye (1991) proved that the best scheme in the traditional EDE sense is no longer the optimal method when the initial-boundary conditions of a hyperbolic problem are discontinuous.

The foregoing discussion motivates the search for a theoretical analysis of errors in hyperbolic problems in which the initial and/or boundary conditions may have discontinuous second derivatives. The present paper develops such a formulation for the water-hammer model.

## EQUIVALENT HYPERBOLIC DIFFERENTIAL EQUATIONS

If both the friction and the convective terms are negligible (as is typically assumed in error analysis), the characteristic form of the water-hammer equations is (Wylie and Streeter 1982)

$$\frac{dH}{dt} \pm \frac{c}{gA} \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \dot{x} = \pm c \quad (1)$$

in which,  $H = H(x, t)$  = piezometric head;  $Q = Q(x, t)$  = volumetric flow rate;  $c$  = physical value of the celerity of the shock wave;  $A$  = cross-sectional area of the pipe;  $g$  = acceleration due to gravity;  $x$  = distance along the pipe's centerline;  $t$  = time; and the  $\pm$  notation is used to distinguish the positive and negative characteristics. As a result of the stated assumptions, these equations are nondissipative and nondispersive.

Hyperbolic partial differential equations, such as the water-hammer equations, faithfully propagate discontinuities in the initial and/or boundary conditions to internal sections (Guenther and Lee 1988). However, the possibility of discontinuity in the solution prevents both the expansion of the dependent variables in a Taylor series and the construction of conventional EDEs. Thus, a new approach is needed.

In essence, EHDEs are formulated by answering the question, what is the system of equations that when integrated results in the same system of algebraic equations as are obtained by interpolating the fixed-grid MOC? More specifically, the construction proceeds as follows. First, the hyperbolic-like terms are collected together. Second, the antiderivative concept is used to convert the algebraic expressions into integral equations. Third, because the latter equations are integrated both with respect to  $x$  and  $t$ , a transformation of variables from  $x$  to  $t$  or vice versa must be invoked. The proper transformation is given by  $x(t)$  such that  $\dot{x}$  equals the propagation speed of

the numerical information. Finally, a limit argument shows that the integrand is zero. It must be emphasized that this formulation does not presume that the dependent variables are analytic. In fact, the EHDE has the same smoothness requirement as the momentum and continuity equations (i.e., the terms  $Q_t$ ,  $H_t$ ,  $Q_x$ , and  $H_x$  must be defined, where the subscripts indicate partial differentiation with respect to  $t$  and  $x$ ). The remainder of the present paper concentrates on constructing EHDE for the wave-speed adjustment approach and for the following interpolation techniques: space-line, reach-out-in-space, time-line, reach-back-in-time, and the Holly-Preissmann scheme. The properties of these EHDEs provide significant insight into how the various discretization approximations alter the physical model.

### **EHDE of Wave Speed Adjustment Method**

The EHDE is derived by artificially changing the wavespeed in (1) from  $c$  to  $\bar{c}$ , thus allowing the grid points to coincide with the intersection of the characteristic curves. The resulting system, which differs both physically and mathematically from the original system, is as follows:

$$\frac{dH}{dt} \pm \frac{\bar{c}}{gA} \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} \quad (2)$$

where  $\bar{c} = c/C_r$ ,  $C_r = c\Delta t/\Delta x =$  Courant number;  $\Delta t =$  time step; and  $\Delta x =$  distance step. Because the wave speed is artificially altered, (2) governs the response of a slightly different pipe system than the one originally considered, and the method is often rejected for this reason. Yet, as the following sections show, the adjusted wave-speed term  $\bar{c} = \Delta x/\Delta t$  appears frequently in the EHDEs for a number of the "pure" interpolation approaches.

The new value of wave speed may be thought of as an adjustment of one or more of the pipe parameters such as the wall thickness, elastic modulus, or fluid density. For example, Lavooij and Tijsseling (1991) viewed the wave-speed adjustment as a modification in density. The adjusted wave speed causes the pulse waves to artificially travel either faster or slower than the actual speed, thereby altering the timing of interaction and reflection of pulses in the system. However, the wave-speed adjustment method is nondissipative and nondispersive, and conserves the total energy in the system (Ghidaoui and Karney 1992).

### **EHDE for Space-Line Interpolation**

Space-line interpolation approximates the solution to the fixed-grid MOC by linearly interpolating the dependent variables of  $Q$  and  $H$  with distance  $x$ . Although the procedure is computationally simple, various researchers have recognized that this approach introduces numerical dissipation and dispersion into the solution (Wiggert and Sundquist 1977; Wylie 1980; Goldberg and Wylie 1983; Lai 1989). The present paper provides a theoretical investigation of the origins of these numerical errors using the EHDE approach.

The space-line interpolation solution of the positive characteristic in (1) is given by linearly interpolating from adjacent grid locations to the point where the characteristic intersects the previous time level, as shown in Fig. 1. [In Fig. 1 the numbered points indicate the interpolated point based on adjacent grid values: point 1 for space-line interpolation, point 2 for time-line interpolation (or reach-back-in-time with a reach-back number of  $j = 1$ ), and point 3 for reach-out-in-space interpolation (reach-out number  $m$

ded from ascelibrary.org by Toronto University Of on 10/04/12. For personal use only. No other uses without permission. Copyright (c) 2012. American Society of Civil Engineers. All rights reserved.

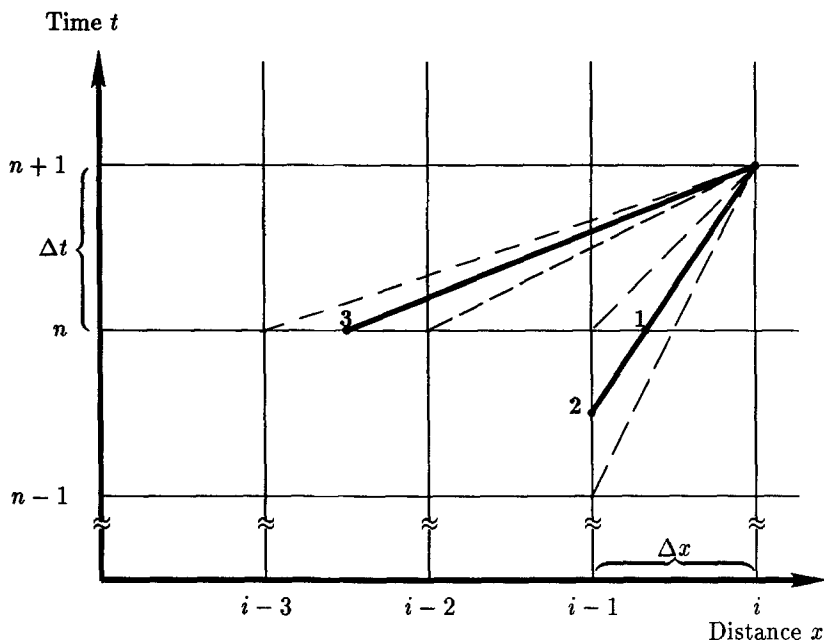


FIG. 1. Interpolation Schemes Illustrated for  $C^+$  Characteristic

= 2). The solid diagonal lines represent the characteristic lines (the path of propagation for the physical wave); the dashed diagonal lines represent the path of the numerical wave based on the associated interpolation scheme.] Stated mathematically, the space-line interpolation scheme can be written

$$H_i^{n+1} - [C_r H_{i-1}^n + (1 - C_r) H_i^n] + B\{Q_i^{n+1} - [C_r Q_{i-1}^n + (1 - C_r) Q_i^n]\} = 0 \quad (3)$$

in which the constant  $B = c/gA$ ; the subscript  $i$  and the superscript  $n$  indicate the spatial and temporal location of the current grid point, respectively; and the expressions in brackets summarize the linear interpolation in terms of  $C_r$ . Eq. (3) can be more conveniently written

$$H_i^{n+1} - H_i^n + C_r(H_i^n - H_{i-1}^n) + B[Q_i^{n+1} - Q_i^n + C_r(Q_i^n - Q_{i-1}^n)] = 0 \quad (4)$$

The expressions  $(H_i^{n+1} - H_i^n)$  and  $(Q_i^{n+1} - Q_i^n)$  are the antiderivatives of  $H_t$  and  $Q_t$ , respectively, where the limits of integration are  $t^n$  and  $t^{n+1}$ , while  $(H_i^n - H_{i-1}^n)$  and  $(Q_i^n - Q_{i-1}^n)$  are the antiderivatives of  $H_x$  and  $Q_x$ , respectively, where the limits of integration are  $x_{i-1}$  and  $x_i$ . Therefore (4) has the following integral equation form:

$$\int_{t^n}^{t^{n+1}} \frac{\partial H}{\partial t} dt + C_r \int_{x_{i-1}}^{x_i} \frac{\partial H}{\partial x} dx + B \left( \int_{t^n}^{t^{n+1}} \frac{\partial Q}{\partial t} dt + C_r \int_{x_{i-1}}^{x_i} \frac{\partial Q}{\partial x} dx \right) = 0 \quad (5)$$

Transforming the variable of integration in the second and fourth terms of (5) from space  $x$  to time  $t$  and recognizing that the expression that maps the interval  $[x_{i-1}, x_i]$  into the interval  $[t^n, t^{n+1}]$  is  $\dot{x} = \bar{c} = c/C_r$ , allows (5) to be reduced as follows:

$$\int_{t^n}^{t^{n+1}} \left[ \frac{\partial H}{\partial t} + C_r \bar{c} \frac{\partial H}{\partial x} + B \left( \frac{\partial Q}{\partial t} + C_r \bar{c} \frac{\partial Q}{\partial x} \right) \right] dt = 0 \quad (6)$$

Note that this equation applies if and only if  $\dot{x} = \bar{c}$ ; however, due to space constraints, this restriction is not continually repeated in the following arguments.

Since (6) is valid for any time step (i.e., for all  $t^{n+1} \geq t^n$ ), the integrand must be identically equal to zero, thus allowing it to be rewritten in a slightly different form as

$$\frac{\partial H}{\partial t} + \bar{c} \frac{\partial H}{\partial x} - (1 - C_r) \bar{c} \frac{\partial H}{\partial x} + B \left[ \frac{\partial Q}{\partial t} + \bar{c} \frac{\partial Q}{\partial x} - (1 - C_r) \bar{c} \frac{\partial Q}{\partial x} \right] = 0 \quad (7)$$

Now, applying the total derivative concept reduces (7) to the following ordinary differential equation:

$$\frac{dH}{dt} + B \frac{dQ}{dt} - (1 - C_r) \bar{c} \left( \frac{\partial H}{\partial x} + B \frac{\partial Q}{\partial x} \right) = 0 \quad (8)$$

Similarly, one derives an EHDE along the negative characteristic line. The equivalent full system of equations is as follows:

$$\frac{dH}{dt} \pm B \frac{dQ}{dt} - \frac{c(1 - C_r)}{C_r} \left\{ \pm \frac{\partial H}{\partial x} + B \frac{\partial Q}{\partial x} \right\} = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} = \pm \frac{c}{C_r} \quad (9)$$

Thus, the EHDE shows that space-line interpolation alters the wave speed from  $c$  to  $\bar{c} = c/C_r$  and introduces a diffusion term with numerical viscosity  $c(1 - C_r)/C_r$ . That is, the MOC with space-line interpolation is equivalent to a wave-diffusion model. Note that the diffusion term goes to zero if  $C_r = 1$ . In addition, the numerical wave speed of space-line interpolation is generally equal to that used in the wave-speed adjustment technique (e.g., if  $C_r = 0.5$ , then the numerical wavespeed  $\bar{c}$  is twice the physical wave speed in both models). However, although the two methods may have the same wave paths, their dynamic models are different.

The diffusion terms involve the first derivative of head and flow with respect to  $x$ . Therefore, these terms are zero for a step wave both upstream and downstream of the discontinuity. Hence, there is no numerical attenuation upstream and downstream of the shock front, but the wave is moving with the adjusted wave speed. At the discontinuity, however, the wave peak is attenuated according to the numerical diffusion coefficient  $c(1 - C_r)/C_r$  and is dispersed according to the spatial gradient of  $H$  and  $Q$ . For instance, the head change following a flow stoppage is theoretically the potential surge. However, due to interpolation the EHDE involves a nonzero flow gradient term at the wave front, which means that the fluid's velocity is not zero and thereby the magnitude of the head is attenuated. This phenomenon is similar to the physical attenuation that occurs in long conduits or in

conduits where friction is high, with the difference being that there is no numerical line packing when  $C_r \leq 1$ . In fact, a large value of  $c(1 - C_r)/C_r$  attenuates the head and flow similarly to a pipeline with a large friction factor. On the other hand, a large flow and head gradient simulates the attenuation similarly to a long pipeline. In addition, when  $C_r > 1$ , the numerical viscosity becomes negative, which is physically unrealistic and confirms Courant's stability criteria. Actually what happens in this case is a numerical wave packing—usually called numerical amplification—by analogy with the physical phenomena of wave packing.

Additional attenuation is created by premature wave reflection at the boundaries. For example, if  $C_r = 0.5$  then the  $\bar{c} = 2c$ . This means that a fraction of the pulse wave generated at, say, the downstream boundary arrives at the upstream boundary in half the time it takes for the physical wave, confirming Wylie and Streeter's [(1982) page 59] heuristic analysis. As a result, this reflected wave interacts with incoming waves and tends to reduce their magnitude.

The foregoing discussion makes it clear that space-line interpolation changes the region of influence, domain of dependence, and the dynamics of the problem. That is, space-line interpolation fundamentally alters the physical model.

### **EHDE for Reach Out in Space Interpolation**

If the foot of the characteristic intersects the time grid  $j$  times before it intersects the space axis, the resulting procedure is called the reach-out-in-space interpolation with a reach-out number of  $j$  (Wiggert and Sunquist 1978; Goldberg and Wylie 1983; Lai 1989). If the MOC is used to solve for head and flow at node  $(n + 1, i)$  then the values of the dependent variables at the foot of the characteristic line are required. These values are determined by linearly interpolating between nodes  $(n, i - j)$  and  $(n, i - j - 1)$  for the positive characteristic line (see Fig. 1 for the case of  $j = 2$ ) and  $(n, i + j)$  and  $(n, i + j + 1)$  for the negative characteristic line. For example, integrating (1) by the MOC and using a reach-out number of  $j$  results in

$$\begin{aligned}
 H_i^{n+1} - \{[(1 + j)C_{rj} - j]H_{i-j-1}^n + [1 + j - (1 + j)C_{rj}]H_{i-j}^n\} \\
 + B(Q_i^{n+1} - \{(1 + j)C_{rj} - j\}Q_{i-j-1}^n \\
 + [1 + j - (1 + j)C_{rj}]Q_{i-j}^n) = 0 \quad (10)
 \end{aligned}$$

where  $C_{rj} = c\Delta t/(1 + j)\Delta x$  is the Courant number. Eq. (10) can be rewritten

$$\begin{aligned}
 [(1 + j)C_{rj} - j][H_i^{n+1} - H_{i-j-1}^n + B(Q_i^{n+1} - Q_{i-j-1}^n)] + [1 + j \\
 - (1 + j)C_{rj}][H_i^{n+1} - H_{i-j}^n + B(Q_i^{n+1} - Q_{i-j}^n)] = 0 \quad (11)
 \end{aligned}$$

If  $\dot{x} = c/C_{rj}$  then

$$H_i^{n+1} - H_{i-j-1}^n = \int_{t^n}^{t^{n+1}} \frac{dH}{dt} dt \text{ and } Q_i^{n+1} - Q_{i-j-1}^n = \int_{t^n}^{t^{n+1}} \frac{dQ}{dt} dt \quad (12a,b)$$

and if  $\dot{x} = jc/(1 + j)C_{rj}$  then

$$H_i^{n+1} - H_{i-j}^n = \int_{t^n}^{t^{n+1}} \frac{dH}{dt} dt \text{ and } Q_i^{n+1} - Q_{i-j}^n = \int_{t^n}^{t^{n+1}} \frac{dQ}{dt} dt \quad (13a,b)$$

Eq. (11) can be written in an integral form by using (12) and (13) and the result transformed into a differential equation by taking the limit as  $t$  tends to zero to give

$$[(1 + j)C_{rj} - j] \left( \frac{dH}{dt} + B \frac{dQ}{dt} \right)_{x,t} + (1 + j)(1 - C_{rj}) \left( \frac{dH}{dt} + B \frac{dQ}{dt} \right)_{u,t} = 0 \quad (14)$$

$$\text{if } \frac{dx}{dt} = \frac{c}{C_{rj}} \quad \text{and} \quad \frac{du}{dt} = \frac{jc}{(1 + j)C_{rj}} \quad (15a,b)$$

in which  $u = x + \Delta x - ct/(j + 1)C_{rj}$  and the subscripts on the parentheses indicate which characteristic equation applies. Note that the term  $\Delta x - ct/(j + 1)C_{rj}$  represents the distance between the leading wave and the trailing wave.

Eq. (14) is the EHDE along the positive characteristic line. Similarly, one derives an EHDE along the negative characteristic line. The EHDE for reach out in space is thus a superposition of two waves, one starting at nodes  $(n, i - j - 1)$  and  $(n, i + j + 1)$  and the other at  $(n, i - j)$  and  $(n, i + j)$ . Actually, if one considers the linear nature of this scheme and the way in which information is propagated from grid points as sketched in Fig. 1, this result is intuitively appealing. Mathematically, the EHDE can be rewritten

$$\frac{dH}{dt} \pm B \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} = \pm \frac{c}{C_{rj}} \quad (16)$$

and

$$\frac{dH}{dt} \pm B \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} = \pm \frac{jc}{(j + 1)C_{rj}} \quad (17)$$

Let the solution of (16) at node  $(n + 1, i)$  be  $(\bar{q})_i^{n+1}$  and  $(\bar{h})_i^{n+1}$  and that of (17) at the same node be  $(\tilde{q})_i^{n+1}$  and  $(\tilde{h})_i^{n+1}$ . Then, the actual solution can be written as the following two equations:

$$Q_i^{n+1} = [(1 + j)C_{rj} - j](\bar{q})_i^{n+1} + [1 + j - (1 + j)C_{rj}](\tilde{q})_i^{n+1} \quad (18)$$

and

$$H_i^{n+1} = [(1 + j)C_{rj} - j](\bar{h})_i^{n+1} + [1 + j - (1 + j)C_{rj}](\tilde{h})_i^{n+1} \quad (19)$$

Eqs. (16) and (17) show that the reach-out-in-space interpolation changes the physical and mathematical models. Both the region of influence and the domain of dependence are altered, thus confirming that this interpolation is dispersive. In addition, the physical model is artificially changed from a single water-hammer wave to a superposition of two waves. This superposition gives insight into the dissipative characteristics of the reach-out scheme. For example, it is readily verified that:  $Q_i^{n+1} \leq \max((\bar{q})_i^{n+1}, (\tilde{q})_i^{n+1})$  and  $H_i^{n+1} \leq \max((\bar{h})_i^{n+1}, (\tilde{h})_i^{n+1})$  if  $C_{rj} \leq 1$ . Now suppose  $\max((\bar{q})_i^{n+1}, (\tilde{q})_i^{n+1}) = (\bar{q})_i^{n+1}$ . Then it is clear from (18) that

Downloaded from ascelibrary.org by Toronto University Of on 10/04/12. For personal use only. No other uses without permission. Copyright (c) 2012, American Society of Civil Engineers. All rights reserved.



$$Q_i^{n+1} \leq [(1 + j)C_{rj} - j](\bar{q})_i^{n+1} + [1 + j - (1 + j)C_{rj}](\bar{q})_i^{n+1} = (\bar{q})_i^{n+1} \quad (20)$$

A similar proof can be made for the head. Hence, the wave peak is attenuated. Further, the trailing wave travels with a speed of  $c/C_{rj}$  that is larger than the physical wave speed; therefore, premature interaction with the boundary conditions usually causes additional attenuation.

If  $C_{rj} > 1$ , the trailing wave is numerically amplified and the leading wave is forced to change sign. This nonphysical amplification and reflection causes the difference equation to diverge from the exact solution to the EHDE. That is, the scheme becomes unstable.

Since the EHDE for reach-out-in-space interpolation is essentially a superposition of two water-hammer problems, the solution of each wave problem can be obtained by the MOC without interpolation, and then the two solutions weighted as indicated by (18) and (19). Hence, at least for the frictionless case, a dedicated reach-out-in-space subroutine is not essential.

The EHDE for reach out in space has the generic form of a fluid-structure interaction (FSI), such as that modeled by Lavooij and Tijsseling (1991). They assumed that the water-hammer equations govern the fluid dynamics while beam theory governs the structural dynamics, each having a different wave speed. Thus, in the sense of speed of propagation, (16) is analogous to the latter phenomena and (17) to the former. Moreover, the weights in (19) perform a similar coupling function as the Poisson's ratio and friction in the FSI model. This analogy makes it clear that reach-out-in-space interpolation has fundamentally changed the physical model.

### EHDE for Time-Line Interpolation

In time-line interpolation, the characteristics are extended through a complete  $\Delta x$  reach and the dependent variables are interpolated between adjacent grid points, as shown in Fig. 1. Thus, integrating (1) using this method yields the following algebraic expression:

$$H_i^{n+1} - [(1 - \xi)H_{i-1}^n + \xi H_{i-1}^{n-1}] + B\{Q_i^{n+1} - [(1 - \xi)Q_{i-1}^n + \xi Q_{i-1}^{n-1}]\} = 0 \quad (21)$$

in which  $\xi = (1 - C_r)/C_r =$  degree of interpolation in time;  $C_r = c\Delta t/\Delta x =$  Courant number, and the expressions in brackets define the interpolation. Simple algebra allows (21) to be rewritten

$$(1 - \xi)(H_i^{n+1} - H_i^n + H_i^n - H_{i-1}^n) + \xi(H_i^{n+1} - H_i^{n-1} + H_i^{n-1} - H_{i-1}^{n-1}) + (1 - \xi)B(Q_i^{n+1} - Q_i^n + Q_i^n - Q_{i-1}^n) + \xi B(Q_i^{n+1} - Q_i^{n-1} + Q_i^{n-1} - Q_{i-1}^{n-1}) = 0 \quad (22)$$

Some helpful identities are now listed

$$y_i^{n+1} - y_i^n = \int_{t^n}^{t^{n+1}} \frac{\partial y}{\partial t} dt \quad (23)$$

$$y_i^{n+1} - y_i^{n-1} = \int_{t^{n-1}}^{t^{n+1}} \frac{\partial y}{\partial t} dt \quad (24)$$

$$y_i^n - y_{i-1}^n = \int_{t^n}^{t^{n+1}} \frac{\partial y}{\partial x} \frac{dx}{dt} dt; \quad \text{if } \frac{dx}{dt} = \bar{c} \quad (25)$$

$$y_i^{n-1} - y_{i-1}^{n-1} = \int_{t^{n-1}}^{t^{n+1}} \frac{\partial y}{\partial x} \frac{dx}{dt} dt; \quad \text{if } \frac{dx}{dt} = \bar{c} \quad (26)$$

where  $t^{n-1} = t^n - \Delta t$  and the variable  $y$  can be either  $Q$  or  $H$ . These identities allow (22) to be rewritten

$$(1 - \xi) \left( \int_{t^n}^{t^{n+1}} \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} \frac{dx}{dt} dt \right) + \xi \left( \int_{t^{n-1}}^{t^{n+1}} \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x} \frac{dx}{ds} ds \right) \\ + (1 - \xi)B \left( \int_{t^n}^{t^{n+1}} \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} \frac{dx}{dt} dt \right) + \xi B \left( \int_{t^{n-1}}^{t^{n+1}} \frac{\partial Q}{\partial s} + \frac{\partial Q}{\partial x} \frac{dx}{ds} ds \right) = 0 \quad (27)$$

To simplify (27), a transformation of variables is required for those terms where the lower limit of integration is  $t^{n-1}$ . Consider  $u$  as the new variable given by the expression  $s = 2u - t^n - \Delta t$  leading to  $ds = 2du$ . The convenience of this transformation stems from the fact that at  $s = t^{n-1} = t^n - \Delta t$ ,  $u = t^n$  and at  $s = t^{n+1}$ ,  $u = t^{n+1}$ . As a result, all the integrands in (27) can now have the same limits of integration; thereby, one can use the identity that the sum of integrands is equal to the integrand of the sum. Eq. (27) can now be rewritten in a form that is more amenable

$$(1 - \xi) \left[ \int_{t^n}^{t^{n+1}} \left( \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} \frac{dx}{dt} \right) dt \right] + \xi \left[ \int_{t^n}^{t^{n+1}} \left( \frac{\partial H(u)}{\partial s} + \frac{\partial H(u)}{\partial x} \frac{dx}{ds} \right) 2du \right] \\ + (1 - \xi)B \left[ \int_{t^n}^{t^{n+1}} \left( \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} \frac{dx}{dt} \right) dt \right] \\ + \xi B \left[ \int_{t^n}^{t^{n+1}} \left( \frac{\partial Q(u)}{\partial s} + \frac{\partial Q(u)}{\partial x} \frac{dx}{ds} \right) 2du \right] = 0 \quad (28)$$

$$\text{if } \frac{dx}{dt} = \bar{c} \quad \text{and} \quad \frac{dx}{du} = \bar{c} \quad (29a,b)$$

Applying the total derivative concept and realizing that the sum of the expression inside the integrands must itself be equal to zero since the integral equation is valid for all  $t^{n+1} \geq t^n$  gives

$$(1 - \xi) \left( \frac{dH}{dt} + B \frac{dQ}{dt} \right) + 2\xi \left( \frac{dH(u)}{ds} + B \frac{dQ(u)}{ds} \right) = 0 \quad (30)$$

$$\text{if } \frac{dx}{dt} = \bar{c} \quad \text{and} \quad \frac{dx}{du} = \bar{c} \quad (31a,b)$$

Time-line interpolation EHDE is a superposition of two waves with different wave speeds. The speeds  $\dot{x} = \bar{c}$  and  $dx/du = \bar{c}$  corresponds to the information propagating from node  $(n, i - 1)$  and node  $(n - 1, i - 1)$ , respectively. In fact, the latter system of equations can be written as follows:

$$\frac{dH}{dt} + B \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \bar{c} \quad (32)$$

and

$$\frac{dH}{dt} + B \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \bar{c} \quad (33)$$

such that the solution for  $H$  and  $Q$  at any grid point is the weighted average of the solutions of (32) and (33). The full EHDE is

$$\frac{dH}{dt} \pm B \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} = \pm \frac{c}{C_r} \quad (34)$$

and

$$\frac{dH}{dt} \pm B \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} = \pm \frac{c}{2C_r} \quad (35)$$

Denoting the solutions of (34) and of (35) at node  $(n + 1, i)$  by  $(\bar{q})_i^{n+1}$ ,  $(\bar{h})_i^{n+1}$  and  $(\tilde{q})_i^{n+1}$ ,  $(\tilde{h})_i^{n+1}$ , respectively, the overall solution is

$$Q_i^{n+1} = (1 - \xi)(\bar{q})_i^{n+1} + \xi(\tilde{q})_i^{n+1} \quad (36)$$

and

$$H_i^{n+1} = (1 - \xi)(\bar{h})_i^{n+1} + \xi(\tilde{h})_i^{n+1}. \quad (37)$$

The expression (37) states that the time-line interpolation solution can be obtained by solving two independent waves at each time level and then using a weighted average of the solution. Thus, as was the case for the reach out in space, the programmer no longer strictly requires a dedicated time-line interpolation subroutine for the frictionless case; instead, the MOC solution can be used twice to independently solve for the two waves and then these solutions summed using  $(1 - \xi)$  and  $(\xi)$  as weights. From the theoretical point of view this result (1) Shows that time-line interpolation alters the physics of the problem—an argument sometimes used by practitioners to dismiss the wave-speed adjustment approach; (2) clearly explains the dispersive property of time-line interpolation; (3) reveals the dissipative characteristic of the scheme; and (4) confirms Courant's linear stability condition. Perhaps points three and four are not obvious and deserve elaboration. Time-line interpolation is dissipative because the weighted average of two wave peaks results in a wave that is lower in magnitude than the original two peaks—the proof of this argument proceeds in the same manner as for reach out in space. When  $C_r > 1$ , one of the weights is larger than 1 while the other is negative. Hence, the wave corresponding to the former weight is amplified and that corresponding to the latter wave changes sign. Both these phenomenon are nonphysical because there is no mechanism that can account for the amplification of the trailing wave, nor are there any boundaries at internal sections that could account for the negative reflection of the leading wave. Therefore, the EHDE confirms the linear stability requirement of Courant.

Additional attenuation is possible due to premature interaction with the boundary condition. For example when  $C_r = 2/3$  then the wave speed of the leading wave is 1.5 times that of the physical wave. That is, the numerical wave arrives at the boundaries three times although in reality it should have arrived only twice. Consequently, the work done at the boundaries is altered, which, by necessity, must alter all the other energy components.

### EHDE for Reach-back-in-Time Interpolation

The reach-back-in-time interpolation with reach-back number  $m$  is based on discretizing the domain in such a way that the characteristic line intersects

the space axis  $m$  times before its foot falls on the time axis [Goldberg and Wylie 1983; Yang and Hsu 1990]. The case of  $m = 1$  is illustrated in Fig. 1. (Although, by convention,  $m \geq 2$  for the reach-back-in-time interpolation, in fact conventional time-line interpolation can be recovered from this more general case by setting  $m = 1$ .) Let  $\xi_m$  and  $\Delta t_m$  represent the temporal interpolation and the time step when the number of reachbacks is  $m$ , respectively. Then the relationship between  $\xi_m$  and  $\Delta t_m$  is (Goldberg and Wylie 1983)

$$\frac{1}{m + \xi_m} = \frac{c\Delta t_m}{\Delta x} \quad (38)$$

If  $C_{rm} = c\Delta t_m/\Delta x =$  the Courant number, then (38) can be rearranged to produce

$$\xi_m = \frac{1 - mC_{rm}}{C_{rm}} \quad (39)$$

The EHDE for reach-back time-line interpolation can be derived in exactly the same manner as that for time-line interpolation by substituting  $\xi_m$  in place of  $\xi$ . The resulting EHDE is

$$\frac{dH}{dt} \pm B \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} = \pm \frac{c}{mC_{rm}} \quad (40)$$

and

$$\frac{dH}{dt} \pm B \frac{dQ}{dt} = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} = \pm \frac{c}{(m + 1)C_{rm}} \quad (41)$$

Let the solution of the leading wave and the trailing wave at node  $(n + 1, i)$  be  $(\bar{q})_i^{n+1}$ ,  $(\bar{h})_i^{n+1}$  and  $(\bar{q})_i^{n+1}$ ,  $(\bar{h})_i^{n+1}$ , respectively. Then, the MOC solution with reach back in time is

$$Q_i^{n+1} = \left(1 - \frac{1 - mC_{rm}}{C_{rm}}\right) (\bar{q})_i^{n+1} + \frac{1 - mC_{rm}}{C_{rm}} (\bar{q})_i^{n+1} \quad (42)$$

and

$$H_i^{n+1} = \left(1 - \frac{1 - mC_{rm}}{C_{rm}}\right) (\bar{h})_i^{n+1} + \frac{1 - mC_{rm}}{C_{rm}} (\bar{h})_i^{n+1} \quad (43)$$

Similar to time-line interpolation, reach-back-in-time interpolation EHDE is a superposition of two waves. Hence, reaching back in time interpolation should not be viewed as a mere numerical convenience; rather, it also is an approximation that changes the physical representation of the problem. Moreover, the computational benefit of the EHDE is realized by recognizing that only the conventional MOC algorithm is required to solve the water-hammer problem. Also, notice that as  $m$  indicates then the two wave speeds (i.e.,  $c/mC_{rm}$  and  $c/(m + 1)C_{rm}$ ) converge to one value. In other words, the physical distortion is minimized by increasing  $m$ , which in turn reduces the numerical errors, confirming the results obtained by Fourier analysis (Goldberg and Wylie 1983). Further, the Courant stability requirement can be deduced from the weights  $(1 - mC_{rm})/C_{rm}$  and  $(1 - mC_{rm})/C_{rm}$ . Indeed, as the effective  $C_{rm}$  exceeds  $1/m$  then the leading wave is amplified in magnitude and the trailing wave changes sign. These two phenomena are nonphysical and lead to unstable results as time increases.

However, increasing  $m$  effectively reduces the time step and increases both the storage and execution requirement. Thus, this option must be compared to using grid refinement along with either the wave-speed adjustment, space-line, or time-line interpolation techniques.

### EHDE for Holly-Preissmann Scheme

To use the Holly-Preissmann scheme, the value of the function and its first derivative must be determined at each computational point (Holly and Preissmann 1977). Holly and Preissmann applied this scheme to the transport equation in two dimensions and have shown, through examples, that their method is superior to the scheme of Leendertse and the scheme of Martin with  $n = 2$ ,  $n = 3$ , and  $n = 4$ . Schohl and Holly (1991) compared the Holly-Preissmann scheme to the fourth-order cubic spline for the pure advection equation. He concluded that the cubic spline is nearly as accurate as the Holly-Preissmann scheme. It should be noted, however, that the cubic spline requires nonphysical boundary conditions. Yang and Hsu (1990) use the time-line version of the Holly-Preissmann scheme. The authors project the foot of the characteristic line to the time-line and use the value of the concentration and its first derivative at the node above and below the interpolation point to determine a third-degree polynomial in time. They showed that this approach is superior to the traditional Holly-Preissmann method in the sense of Fourier mode analysis. Sibetheros et al. (1991) were the first to apply the Holly-Preissmann scheme to the water-hammer problem. They conclude that the Holly-Preissmann method and the cubic spline have similar dissipation and dispersion characteristics with the latter being unconditionally stable. The reported success of the Holly-Preissmann scheme has prompted the present writers to study the theoretical properties of this approach when used in conjunction with the MOC to solve hyperbolic equations.

Using the Holly-Preissmann scheme to evaluate the flow  $Q$  and the head  $H$  between the grid points  $(n, i - 1)$  and  $(n, i)$  gives

$$Q_R = N_1 Q_{i-1}^n + N_2 Q_i^n + N_3 \Delta x \left( \frac{\partial Q}{\partial x} \right)_{i-1}^n + N_4 \Delta x \left( \frac{\partial Q}{\partial x} \right)_i^n \quad (44)$$

and

$$H_R = N_1 H_{i-1}^n + N_2 H_i^n + N_3 \Delta x \left( \frac{\partial H}{\partial x} \right)_{i-1}^n + N_4 \Delta x \left( \frac{\partial H}{\partial x} \right)_i^n \quad (45)$$

where  $N_1, N_2, N_3$  and  $N_4 =$  shape functions defined by the following expressions (Holly and Preissmann 1977):  $N_1 = C_r^2(3 - 2C_r)$ ,  $N_2 = (1 - C_r)(1 + C_r - 2C_r^2)$ ,  $N_3 = C_r^2(1 - C_r)$ , and  $N_4 = -C_r(1 - C_r)^2$ . It can easily be verified that  $N_1 + N_2 = 1$ . The Holly-Preissmann solution of (1) is given by

$$\begin{aligned}
 & H_i^{n+1} - H_{i-1}^n - N_2(H_i^n - H_{i-1}^n) - N_3 \Delta x \left( \frac{\partial H}{\partial x} \right)_{i-1}^n - N_4 \Delta x \left( \frac{\partial H}{\partial x} \right)_i^n \\
 & + B \left[ Q_i^{n+1} - Q_{i-1}^n - N_2(Q_i^n - Q_{i-1}^n) \right. \\
 & \left. - N_3 \Delta x \left( \frac{\partial Q}{\partial x} \right)_{i-1}^n - N_4 \Delta x \left( \frac{\partial Q}{\partial x} \right)_i^n \right] = 0 \quad (46)
 \end{aligned}$$

Let  $Y$  be either  $H$  or  $Q$ , then the antiderivatives for  $(Y_i^{n+1} - Y_{i-1}^n)$ ,  $(Y_i^n - Y_{i-1}^n)$ , and  $\Delta x$  are  $\int_{t_i^n}^{t_i^{n+1}} (dY/dt) dt$ ,  $\int_{t_i^n}^{t_i^{n+1}} \bar{c} (\partial Y/\partial x) dt$ , and  $\int_{t_i^n}^{t_i^{n+1}} \bar{c} dt$ , respectively, if  $\dot{x} = \bar{c}$ . Using these identities to transform (46) into an integral equation and taking the limit as  $\Delta t$  tends to zero yields

$$\begin{aligned}
 \frac{dH}{dt} + B \frac{dQ}{dt} - N_2 \bar{c} \left( \frac{\partial H}{\partial x} + B \frac{\partial Q}{\partial x} \right) - N_3 \bar{c} \left[ \left( \frac{\partial H}{\partial x} \right)_{i-1}^n + B \left( \frac{\partial Q}{\partial x} \right)_{i-1}^n \right] \\
 - N_4 \bar{c} \left[ \left( \frac{\partial H}{\partial x} \right)_i^n + B \left( \frac{\partial Q}{\partial x} \right)_i^n \right] = 0; \quad \text{if } \frac{dx}{dt} = \bar{c} \quad (47)
 \end{aligned}$$

Similar equation can be developed along the negative characteristic line. Thus, the full EHDE is as follows:

$$\begin{aligned}
 \frac{dH}{dt} \pm B \frac{dQ}{dt} - N_2 \bar{c} \left\{ \frac{\partial H}{\partial x} \pm B \frac{\partial Q}{\partial x} \right\} - N_3 \bar{c} \left\{ \left( \frac{\partial H}{\partial x} \right)_{i-1}^n \pm B \left( \frac{\partial Q}{\partial x} \right)_{i-1}^n \right\} \\
 - N_4 \bar{c} \left\{ \left( \frac{\partial H}{\partial x} \right)_i^n \pm B \left( \frac{\partial Q}{\partial x} \right)_i^n \right\} = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} \quad (48)
 \end{aligned}$$

The above system of EHDE represent a wave-diffusion model with source and/or sink terms. In fact, the first two terms involving the total time derivative in each of the dynamic equations constitute the wave equation; the partial derivatives with respect to  $x$  are the diffusive terms; and the partial derivatives at nodes  $(n, i - 1)$  and  $(n, i)$  represent source and/or sink expressions.

The numerical coefficient of viscosity  $N_2$  is physical only if the  $C_r \leq 1$ . Thus, the EHDE gives an independent linear stability analysis.

Notice that the hyperbolic terms in EHDE of space-line interpolation and that of Holly-Preissmann scheme are identical. In addition, the wave speeds of both approaches are identical. However, their viscosity coefficients are different except at  $C_r = 0, 1/2$ , and  $1$ . For  $C_r \in (0, 1/2)$ , the numerical damping coefficient of the Holly-Preissmann scheme is larger than that of the space-line interpolation; therefore, the former scheme is more diffusive than the latter. Conversely, if  $C_r \in (1/2, 1)$ , then the space-line interpolation approach is more diffusive than the Holly-Preissmann scheme. Yet the global diffusive character of the Holly-Preissmann scheme depends, also, on the source and/or sink terms. Therefore, if the nodal derivative terms are sources, then the Holly-Preissmann scheme is definitively superior for  $C_r \in (1/2, 1)$  and, depending on the magnitude of the sources, it may or may not be superior in the rest of the interval. On the other hand, if the nodal derivatives are sinks, then the space-line is definitely superior if  $C_r \in (0, 1/2)$ , and, again depending on the magnitudes of these sinks, it may or may not be superior in the rest of the interval.

One reason that sources and sinks may adversely affect the solution is that the nodal derivatives are only approximately known. The nature and properties of the errors associated with the numerical solution of the nodal derivatives is the subject of the rest of this section.

The solution procedure for determining  $(Q_x)_i^{n+1}$  and  $(H_x)_i^{n+1}$  is numerically determined as follows: the ordinary differential operator governing  $Q_x$  and  $H_x$  is the same as that governing  $Q$  and  $H$ . While the method of characteristics is used to integrate these equations, the derivatives of  $Q_R$  and  $H_R$  are determined by differentiating (44) and (45) at  $x = x_R$ . Mathematically

$$\begin{aligned}
 & \left(\frac{\partial H}{\partial x}\right)_i^{n+1} + \frac{1}{\Delta x} \frac{dN_1}{dC_r} H_{i-1}^n + \frac{1}{\Delta x} \frac{dN_2}{dC_r} H_i^n + \frac{dN_3}{dC_r} \left(\frac{\partial H}{\partial x}\right)_{i-1}^n \\
 & + \frac{dN_4}{dC_r} \left(\frac{\partial H}{\partial x}\right)_i^n + B \left[ \left(\frac{\partial Q}{\partial x}\right)_i^{n+1} + \frac{1}{\Delta x} \frac{dN_1}{dC_r} Q_{i-1}^n + \frac{1}{\Delta x} \frac{dN_2}{dC_r} Q_i^n \right. \\
 & \left. + \frac{dN_3}{dC_r} \left(\frac{\partial Q}{\partial x}\right)_{i-1}^n + \frac{dN_4}{dC_r} \left(\frac{\partial Q}{\partial x}\right)_i^n \right] = 0 \tag{49}
 \end{aligned}$$

where  $dN_1/dC_r = -dN_2/dC_r$ , and thus the EHDE for the system of equations can be written

$$\begin{aligned}
 & \frac{d}{dt} \left(\frac{\partial H}{\partial x}\right) \pm B \frac{d}{dt} \left(\frac{\partial Q}{\partial x}\right) - \frac{\bar{c}}{\Delta x} \frac{dN_1}{dC_r} \left(\frac{\partial H}{\partial x} \pm B \frac{\partial Q}{\partial x}\right) \\
 & + \frac{\Delta x}{\bar{c}} \left(1 + \frac{dN_3}{dC_r}\right) \left[ \left(\frac{\partial H}{\partial x}\right)_{i-1}^n \pm B \left(\frac{\partial Q}{\partial x}\right)_{i-1}^n \right] \\
 & + \frac{\Delta x}{\bar{c}} \frac{dN_4}{dC_r} \left[ \left(\frac{\partial H}{\partial x}\right)_i^n \pm B \left(\frac{\partial Q}{\partial x}\right)_i^n \right] = 0; \quad \text{if } \frac{dx}{dt} = \pm \bar{c} \tag{50}
 \end{aligned}$$

Consequently, the EHDE governing the  $(H_x, Q_x)$  terms constitute a wave-diffusion model with sources and/or sink terms and a numerical wave-speed equal to  $\bar{c}$ . Again, the diffusion coefficient,  $dN_1/dC_r = 6C_r(1 - C_r)$ , yields the Courant condition. Further, the EHDE shows that while the numerical distortion of the wave path is a function of  $C_r$  only, the diffusion and source terms are a function of both  $C_r$  and the wave profile. This dependency on the wave profile shows why the sudden closure case is not necessarily conclusive: for this case, the diffusion and sink terms reduce to zero both upstream and downstream of the discontinuity.

In summary, the EHDE for the Holly-Preissmann scheme shows that computation of  $(H_x, Q_x)$  is prone to numerical errors. In addition, the derivative boundary conditions for most realistic pipeline systems, such as those involving pumps, valves, and other devices, is difficult or impossible to obtain. These numerical and physical difficulties often outweigh the hypothetical benefit associated with the quadratic nature of the Holly-Preissmann.

## CONCLUSIONS

There is a tendency among practitioners to think of interpolation as a numerical device with only numerical side effects. The EHDE approach

described in the present paper shows conclusively that this is not the case. In fact, interpolation fundamentally changes the physical problem and must be viewed as a nontrivial transformation of the governing equations.

The present paper shows that the EHDE for the conventional time-line, reach-back-in-time, and reach-out-in-space interpolations consist of a superposition of two water-hammer problems, each of which has a wave speed different from the physical wave speed. In effect, these techniques change the wave speed twice and then weight the resulting two solutions. The numerical dissipation and dispersion for these schemes arise from the superposition of these nonphysical wave speeds. Moreover, the space-line and the Holly-Preissmann schemes are equivalent to a wave-diffusion problem with an adjusted wave speed, but in addition, the Holly-Preissmann technique involves source and/or sink terms. Further, the fact that all interpolation methods alter the wave speed of the water-hammer model suggests that the simplest of all numerical techniques, that of adjusting the wave speed, should be reconsidered in this light.

The EHDE provides an independent proof for the Courant condition and shows that the physical distortion is a function of both  $C_r$  and the wave profile. This being the case, the sudden closure problem conventionally selected in numerical studies may not always be the most stringent test, since the diffusion and source terms vanish upstream and downstream from the discontinuity. In addition, the EHDE provides a test for consistency because all schemes produce no distortion when  $C_r = 1$ . Moreover, the EHDE suggests that higher-order methods that involve more and more nodal points essentially increase the physical distortion of the problem. For example, using nodes  $i$ ,  $i - 1$ , and  $i - 2$  is equivalent to a superposition of two waves plus diffusion and source terms. It is thus clear that the only way of achieving accurate, general solutions for hyperbolic equations is to keep the time step small and the Courant number as close to 1 as possible.

## APPENDIX I. REFERENCES

- Chaudhry, M. H., and Hussaini, M. Y. (1985). "Second-order accurate explicit finite-difference schemes for waterhammer analysis." *J. Fluid Eng.*, 107(4), 523–529.
- Dammuller, D. C., Bhallamudi, S. M., and Chaudhry, M. H. (1989). "Modelling unsteady flow in curved channel." *J. Hydr. Engrg.*, ASCE, 115(11), 1479–1495.
- Ghidaoui, M. S., and Karney, B. W. (1992). "Time discretization in fixed grid method of characteristics." *Proc., Annu. Conf. of the CSCE*, Canadian Society of Civil Engineers (CSCE), Quebec City, Canada, 367–376.
- Goldberg, D. E., and Wylie, E. B. (1983). "Characteristics method using time-line interpolations." *J. Hydr. Engrg.*, ASCE, 109(5), 670–683.
- Guenther, R. B., and Lee, J. W. (1988). *Partial differential equations of mathematical physics and integral equations*. Prentice-Hall, Englewood Cliffs, N.J.
- Holly, M., and Preissmann, A. (1977). "Accurate calculation of transport in two dimensions." *J. Hydr. Engrg.*, ASCE, 103(11), 1259–1277.
- Katopodes, N. D. (1984). "Fourier analysis of dissipative FEM channel flow model." *J. Hydr. Engrg.*, ASCE, 110(7), 927–944.
- Lai, C. (1989). "Comprehensive method of characteristics models for flow simulation." *J. Hydr. Engrg.*, ASCE, 114(9), 1074–1095.
- Lavoij, C. S. W., and Tijsseling, A. S. (1991). "Fluid-structure interaction in liquid-filled piping systems." *J. Fluids and Struct.*, 5, 573–595.
- Leonard, B. P. (1979). "A stable and accurate convective modelling procedure based on quadratic upstream interpolation." *Comp. Methods in Appl. Mech. and Engrg.*, 19, 59–98.
- Noye, B. J. (1991). "Some three-level finite difference methods for simulating advection in fluids." *Comp. & Fluids*, 19(1), 119–140.



- O'Brian, G. G., Hyman, M. A., and Kaplan, S. (1951). "A study of the numerical solution of partial differential equations." *J. Math. Phys.*, 29(4), 223–251.
- Samuels, G. P., and Skeels, P. C. (1990). "Stability limits for Preissmann's scheme." *J. Hydr. Engrg.*, ASCE, 116(8), 997–1011.
- Schohl, G. A., and Holly, F. M. (1991). "Cubic-spline interpolation in Lagrangian advection computation." *J. Hydr. Engrg.*, ASCE, 117(2), 248–253.
- Sibetheros, I. A., Holley, E. R., and Branski, J. M. (1991). "Spline interpolations for water hammer analysis." *J. Hydr. Engrg.*, ASCE, 117(10), 1332–1349.
- Sivaloganathan, K. (1978). "Flood routing by characteristic methods." *J. Hydr. Div.*, ASCE, 107(7), 1075–1091.
- Wiggert, D. C., and Sundquist, M. J. (1977). "Fixed-grid characteristics for pipeline transients." *J. Hydr. Div.*, ASCE, 103(12), 1403–1415.
- Wylie, E. B. (1980). "Inaccuracies in the characteristics method." *Proc., Spec. Conf. on Comp. and Physical Modelling in Hydr. Engrg.*, ASCE, New York, N. Y., 165–176.
- Wylie, E. B., and Streeter, V. L. (1982). *Fluid transients*. Feb Press, Ann Arbor, Mich.
- Yanenko, N. N., Fedotova, Z. I., Tusheva, L. A., and Shokin, Y. I. (1983). "Classification of difference schemes of gas dynamics by the method of differential approximation—I." *Comp. and Fluids*, 11(3), 187–206.
- Yang, J. C., and Hsu, E. L. (1990). "Time-line interpolation for solution of the dispersion equation." *J. Hydr. Res.*, 28(4), 503–523.

## APPENDIX II. NOTATION

*The following symbols are used in this paper:*

- $A$  = cross-sectional area of pipe;  
 $B$  = pipe constant  $cgA$ ;  
 $C_r$  = Courant number:  $m = 1$  and  $j = 0$ ;  
 $C_{rm}, C_{rj}$  = Courant number:  $m > 1$  and  $j = 0$ ;  $m = 1$  and  $j > 0$ ;  
 $c, (\bar{c}, \bar{\bar{c}})$  = physical wave speed (numerical wave speeds);  
 $g$  = acceleration due to gravity;  
 $H$  = piezometric head:  $H = H(x, t)$ ;  
 $H_R$  = piezometric head at interpolation point  $R$ ;  
 $H_i^n$  = piezometric head at node  $(n, i)$ ;  
 $(\bar{h}), (\bar{\bar{h}})$  = piezometric head corresponding to  $\bar{c}$  and  $\bar{\bar{c}}$ ;  
 $i$  = integer denoting spatial nodal location;  
 $m, j$  = integer denoting number of reaches back, reaches out;  
 $n$  = integer denoting time nodal location;  
 $N_1, N_2, N_3, N_4$  = shape functions for Holly-Preissmann scheme;  
 $Q$  = volumetric flow function:  $Q = Q(x, t)$ ;  
 $Q_R$  = volumetric flow at interpolation point  $R$ ;  
 $Q_i^n$  = volumetric flow at node  $(n, i)$ ;  
 $(\bar{q}), (\bar{\bar{q}})$  = volumetric flow corresponding to  $\bar{c}$  and  $\bar{\bar{c}}$ ;  
 $s$  = time as dummy variable;  
 $t, t^n$  = time, time at  $n\Delta t$ ;  
 $x, \Delta x$  = space coordinate, space increment;  
 $\dot{x}$  = total derivative of  $x$  with  $t$ ;  
 $\Delta t, \Delta t_m$  = time step, time step when  $m > 1$ ;  
 $\xi$  = degree of time-line interpolation; and  
 $\xi_m$  = degree of time-line interpolation when  $m > 1$ .